ITINERIS

Introduction to Lagrangian Statistics

- Joe LaCasce

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Introduction to Lagrangian Statistics

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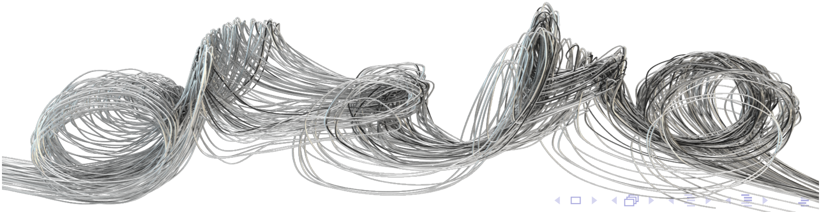
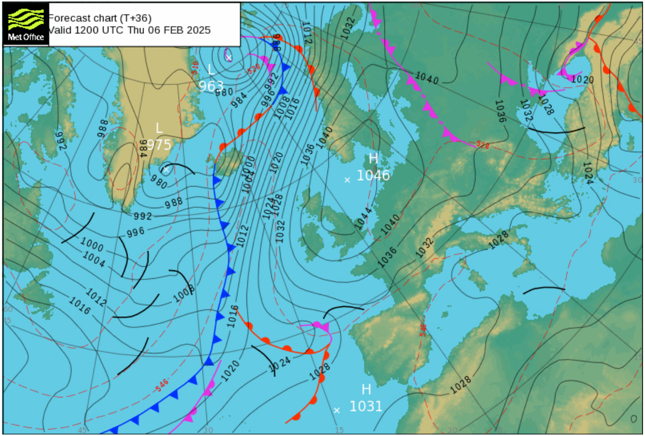
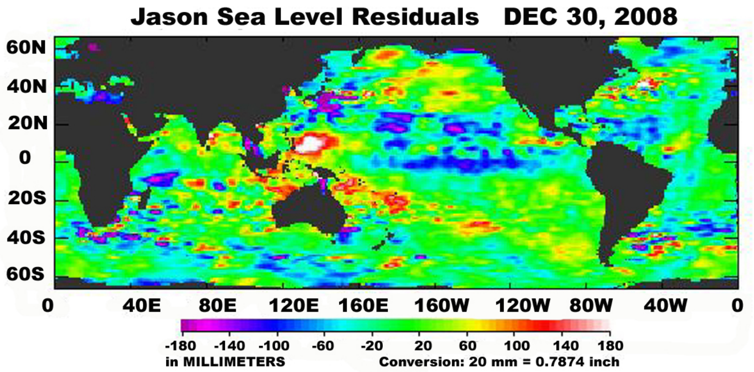


Eulerian data



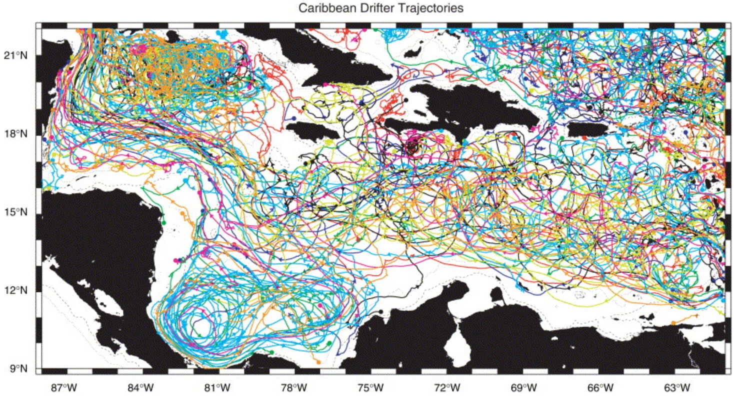
UK Met Office

Eulerian data



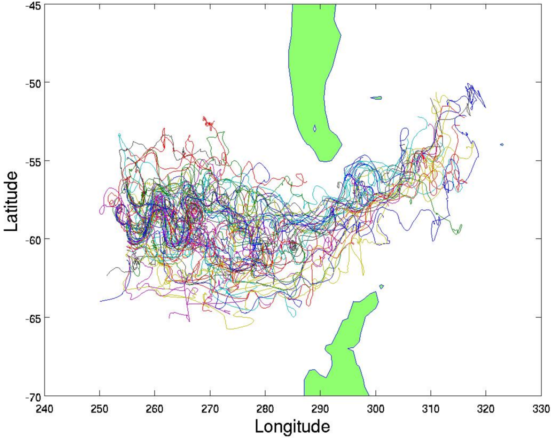
GSFC/NASA

Lagrangian data



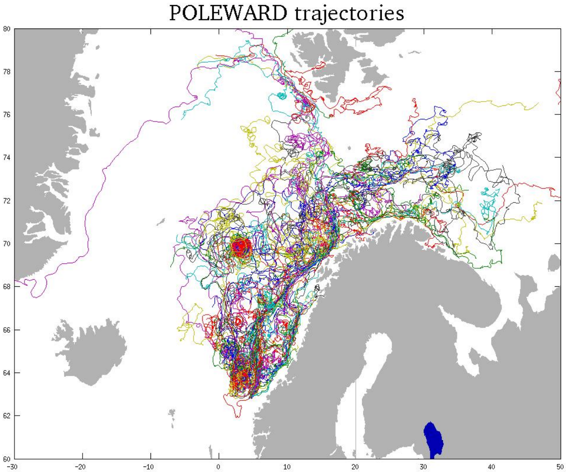
Richardson, 2005

Lagrangian data



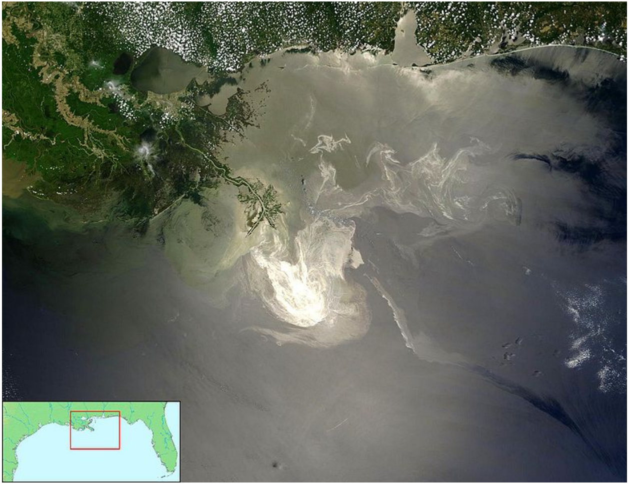
The DIMES float group

Lagrangian data



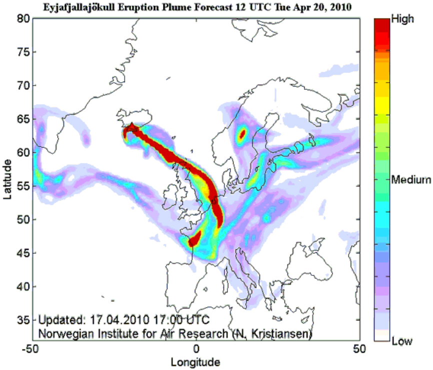
Koszalka et al., 2009

Deepwater Horizon oil spill (2010)



NASA

Icelandic volcano (2010)



R. Sigurdsson, NILU

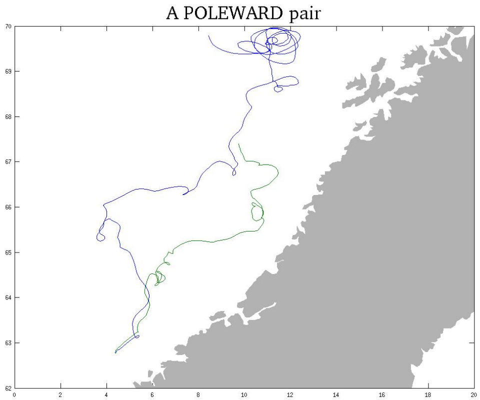
Lagrangian data

What to do with Lagrangian data?

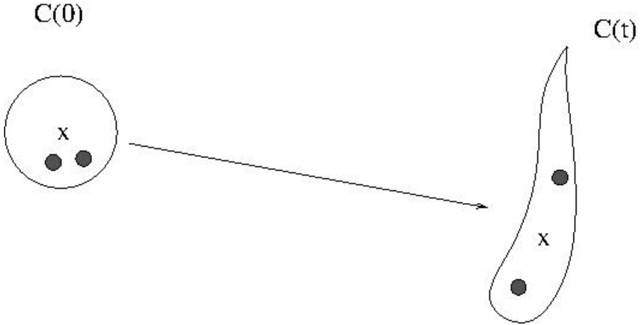
- Descriptive studies
- Data assimilation
- Dynamical systems studies
- **Statistical analysis**

Statistics

- No two trajectories are alike



Tracer spreading



First moment

The mean position of the cloud in x is:

$$\overline{x}(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$$

→ The first moment is a *single particle statistic*

Second moment

The variance in the x-direction is:

$$D_x(t) = \frac{1}{N-1}\sum_{i=1}^{N}(x_i(t)-\overline{x}(t))^2$$

Expanding this, it's possible to rewrite this as:

$$D_x(t) = \frac{1}{2N(N-1)}\sum_{i\neq j}(x_i(t)-x_j(t))^2$$

→ The second moment is a *particle pair statistic*

Outline

I. Single particle statistics

- Random walk, diffusion
- Autocorrelation, Lagrangian time scale
- Diffusivity
- Mean flow effects

II. Particle pair statistics

- Velocity correlation
- Structure functions and spectra
- PDFs
- Relative dispersion, kurtosis

Random walk

Random walk

Consider an idealized drunk. Assume he/she takes uniform steps, of length s, but that each step is randomly oriented:

$$\mathbf{r}_n = \mathbf{r}_{n-1} + \mathbf{s}$$

So:

$$|\mathbf{r}_n|^2 = |\mathbf{r}_{n-1}|^2 + s^2 + 2\mathbf{r}_{n-1} \cdot \mathbf{s}$$

Random walk

Consider a *group* of drunks. Their *dispersion* (mean square displacement) is:

$$\overline{|\mathbf{r}_n|^2} = \overline{|\mathbf{r}_{n-1}|^2} + s^2$$

The overline indicates an *ensemble average*, i.e. the average over all drunks. Assuming the initial displacement is zero:

$$\overline{|\mathbf{r}_1|^2} = 0 + s^2$$

and:

$$\overline{|\mathbf{r}_2|^2} = 2s^2$$

Random walk

Thus:

$$\overline{|\mathbf{r}_n|^2} = ns^2$$

Assume the drunks take one step per second:

$$\overline{|\mathbf{r}_n|^2} \propto s^2 t$$

The dispersion increases *linearly in time*, a characteristic of *Brownian motion* (Einstein, 1905).

Diffusion

The equation for a passive tracer is:

$$\frac{\partial}{\partial t} C + \nabla \cdot (\mathbf{u} C) = \kappa \nabla^2 C$$

where κ is the *diffusivity*. Neglecting advection:

$$\frac{\partial}{\partial t} C = \kappa \nabla^2 C$$

Consider a cloud of tracer (e.g. smoke). The cloud variance is:

$$\overline{\mathbf{r}^2} = \frac{\iint \mathbf{x}^2 C \, dA}{\iint C \, dA}$$

Diffusion

We can derive an equation for $\overline{x^2}$ from the 1-D tracer equation:

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty} x^2 C(x,t)\,dx &= \int_{-\infty}^{\infty} x^2\,\kappa\,\frac{\partial^2}{\partial x^2} C\,dx\\ &= -2\kappa\int_{-\infty}^{\infty} x\frac{\partial}{\partial x} C\,dx\\ &= 2\kappa\int_{-\infty}^{\infty} C\,dx\end{aligned}$$

Diffusion

So:

$$\frac{d}{dt}\overline{x^2} = \frac{d}{dt}\frac{\iint x^2 C \, dx}{\iint C \, dx} = 2\kappa$$

Thus:

$$\overline{x^2} = 2\kappa t$$

The variance increases linearly in time, just like with a random walk → the random walk is a *diffusive process*.

"Eddy diffusion"

On synoptic (i.e. weather) scales, molecular diffusion is *much* weaker than advective stirring. But if the advection is *random*, similar ideas apply.

We can introduce an "eddy diffusivity" and talk about random motion from turbulent advection

Eddy diffusion

Let:

$$\mathbf{u} = \overline{\mathbf{u}} + u', \quad C = \overline{C} + C'$$

with:

$$\overline{\mathbf{u}'} = \overline{C'} = 0$$

Averaging the tracer equation yields:

$$\frac{\partial}{\partial t}\overline{C} + \nabla \cdot (\overline{\mathbf{u}}\,\overline{C}) + \nabla \cdot (\overline{\mathbf{u}' C'}) = \nabla \cdot (\kappa \nabla \overline{C})$$

Eddy diffusion

Often *parameterize* the eddy correlation in terms of a down-gradient flux:

$$\overline{\mathbf{u}' C'} = -K \nabla \overline{C}$$

Then:

$$\frac{\partial}{\partial t}\overline{C} + \nabla \cdot (\overline{\mathbf{u}}\,\overline{C}) = \nabla \cdot ((K + \kappa)\nabla \overline{C})$$

Generally $K \gg \kappa$, so we ignore the latter

But how to determine K?

Diffusion by continuous moments

G. I. Taylor (1921)

Diffusion by continuous moments

Consider a group of particles. We can define:

$$K_x \equiv \frac{1}{2}\frac{d}{dt}\overline{(x-x_0)^2}$$

We can rewrite this:

$$\begin{aligned} K_x &= \overline{u(t)(x-x_0)} \\ &= \overline{u(t)\int_0^t u(t')\,dt'} \\ &= \int_0^t \overline{u(t)u(t')}\,dt' \end{aligned}$$

Diffusion by continuous moments

If the flow is *stationary*:

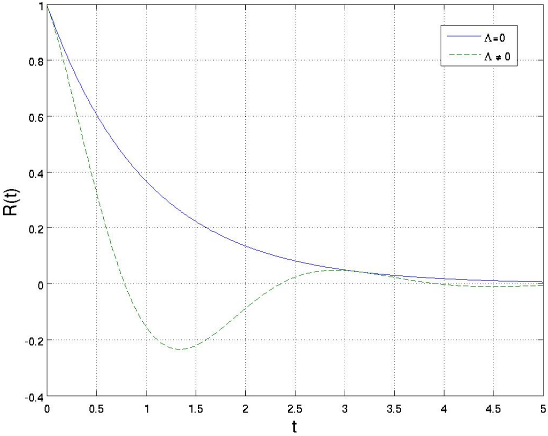
$$K_x = \nu^2 \int_0^t R(t')\, dt'$$

where ν^2 is the velocity variance and

$$R(t) \equiv \frac{1}{\nu^2} \overline{u(t)\, u(t')}$$

is the velocity *autocorrelation*.

Autocorrelation



Asymptotic limits

At early times:

$$R(t) = 1 + \frac{dR}{dt} t + ...$$

So:

$$lim_{t \to 0} \; R = 1$$

Thus:

$$lim_{t \to 0} \; K_x = \nu^2 \int_0^t dt = \nu^2 t$$

Asymptotic limits

At long times, the velocity becomes decorrelated. So the integral of the autocorrelation should converge:

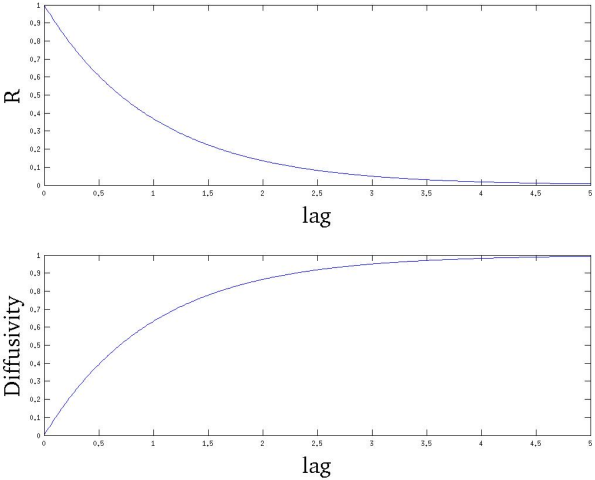
$$T_L = \int_0^\infty R(t')\, dt' = const.$$

This the *Lagrangian integral time*. It is a measure of the predictability of the particle motion. Also, the diffusivity is constant:

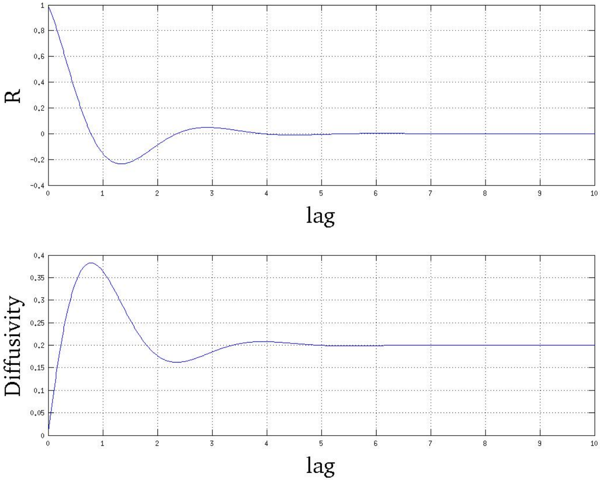
$$lim_{t\rightarrow\infty}\, K = \nu^2 T_L$$

So the system can be modeled using a diffusion equation.

Example: exponential decay



Example: mixed exponential/oscillatory decay



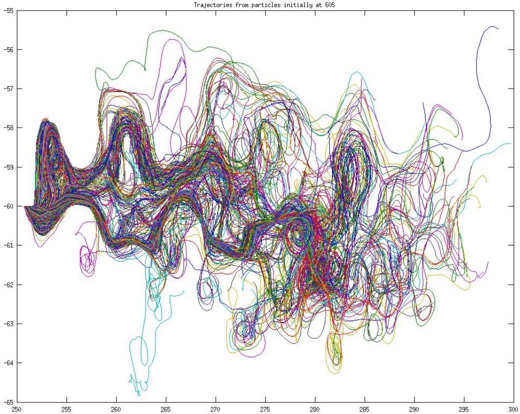
Diffusivity

Three different ways to calculate the diffusivity, here in y:

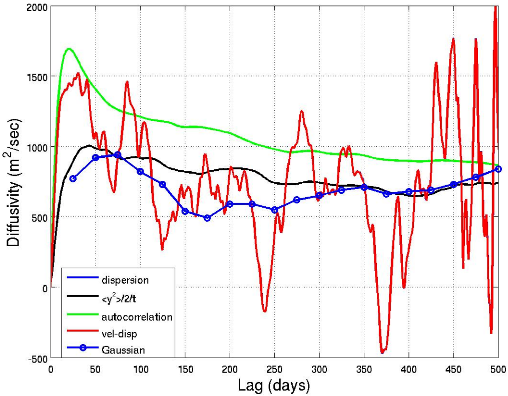
$$K_y(t) = \frac{1}{2}\frac{d}{dt}\overline{(y(t) - y_0)^2}$$
$$= \overline{v(t)(y - y_0)}$$
$$= \nu^2 \int_0^t R(t')\, dt'$$

These typically yield *different results*

Southern Ocean trajectories, MITgcm



Meridional diffusivities



JHL et al. (2014)

Fourth method

If the motion is diffusive, then:

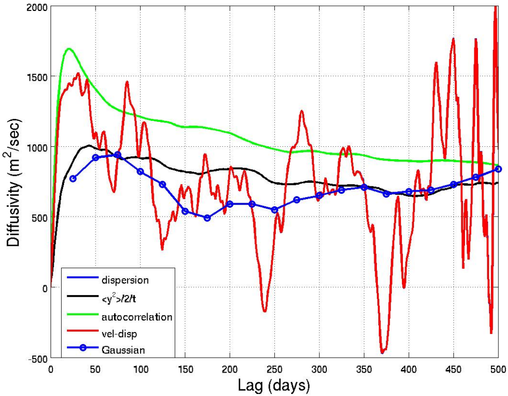
$$\overline{y^2} = 2K_y t$$

So:

$$K_y = \frac{\overline{y^2}}{2t}$$

Perhaps the simplest method

Meridional diffusivities



JHL et al. (2014)

Fifth method

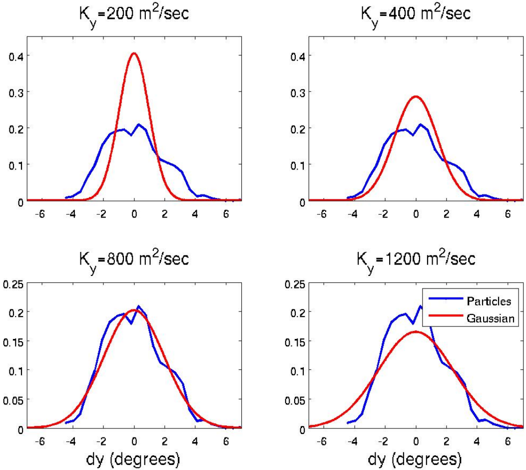
With a constant diffusivity, the probability density function (PDF) of the meridional displacements is:

$$p(y) = \frac{p_0}{2\sqrt{\pi K_y t}} \, exp(-\frac{y^2}{4K_y t})$$

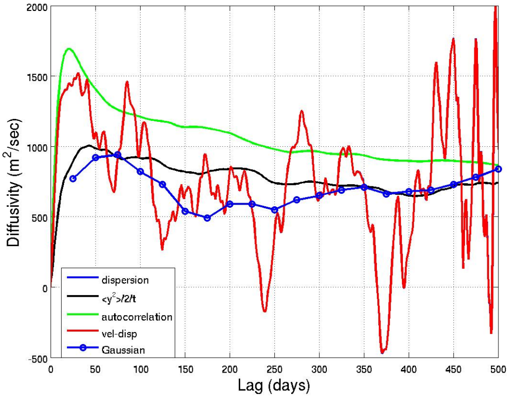
By comparing the actual PDF with this, we can evaluate:

- How “diffusive” the spreading actually is
- A best estimate for K_y

Meridional PDFs after 1 year



Meridional diffusivities



JHL et al. (2014)

Complicating factors

Taylor's method gives us a prescription for how to estimate diffusivities from Lagrangian data

Relevant for spreading of oil, volcanic ash, etc.

But the ocean is **inhomogeneous**, with the eddy kinetic energy varying greatly in space

Also, there are significant **time-mean flows** (e.g. Gulf Stream) and these impact the calculations

Shear dispersion

A mean *shear* in particular can greatly affect the diffusivity. Consider:

$$u = U(y) + u', \qquad |u'| \ll \overline{U}(y)$$

Then the tracer equation is approximately:

$$\frac{\partial}{\partial t}C + U\frac{\partial}{\partial x}C = K\nabla^2 C = K_x\frac{\partial^2}{\partial x^2}C + K_y\frac{\partial^2}{\partial y^2}C$$

(assuming a constant but not isotropic K)

First moment in x

From this we can derive the moments in x. For example:

$$\frac{\partial}{\partial t} xC + xU \frac{\partial}{\partial x} C = xK \nabla^2 C$$

Integrating:

$$\frac{d}{dt} \iint xC \, dA + \iint xU \frac{\partial}{\partial x} C \, dA = \iint xK \nabla^2 C \, dA$$

The diffusive term is zero and second term is:

$$\iint xU \frac{\partial}{\partial x} C \, dA = - \iint UC \, dA$$

Moments in x

So:

$$\frac{d}{dt}\overline{x} = \frac{d}{dt}\frac{\iint xC\,dA}{\iint C\,dA} = \frac{\iint UC\,dA}{\iint C\,dA} \equiv \overline{U}$$

Similarly, can show that the second moment is:

$$\frac{d}{dt}\overline{x^2} = 2\overline{xU} + 2K_x$$

The dispersion is affected by the correlation between U and x

Examples

1) $U = 0$:

$$\overline{x^2} = 2K_x t$$

2) $U = a$

$$\frac{d}{dt}\overline{x} = a \quad \rightarrow \quad \overline{x} = at$$

$$\frac{d}{dt}\overline{x^2} = 2\overline{xU} + 2K_x = 2a^2 t + 2K_x$$

$$\rightarrow \overline{x^2} - \overline{x}^2 = 2K_x t$$

- Constant drift with diffusive spreading

Shear flow

3) $U = a + by$

$$\frac{d}{dt}\overline{x} = a \quad \rightarrow \quad \overline{x} = at$$

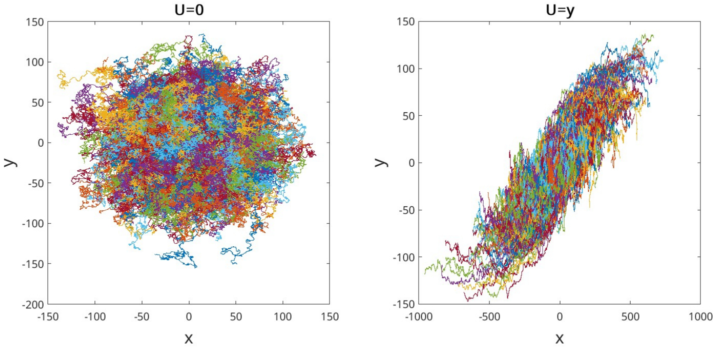
$$\frac{d}{dt}\overline{x^2} = 2\overline{xU} + 2K_x = 2a^2 t + 2b\overline{xy} + 2K_x$$

$$\frac{d}{dt}\overline{x^2} = 2a^2 t + 2b^2 K_y t^2$$

$$\rightarrow \overline{x^2} - \overline{x}^2 = \frac{2}{3} b^2 K_y t^3$$

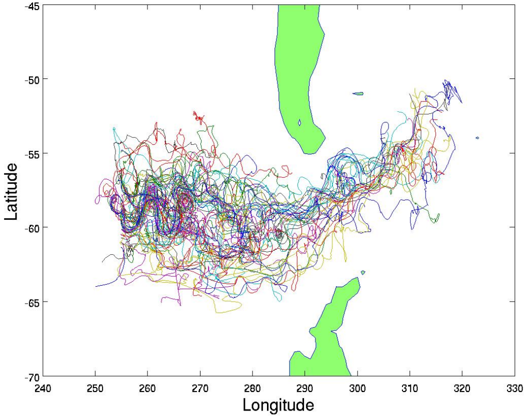
- The zonal diffusivity is *not constant* in time!

Shear dispersion with stochastic spreading



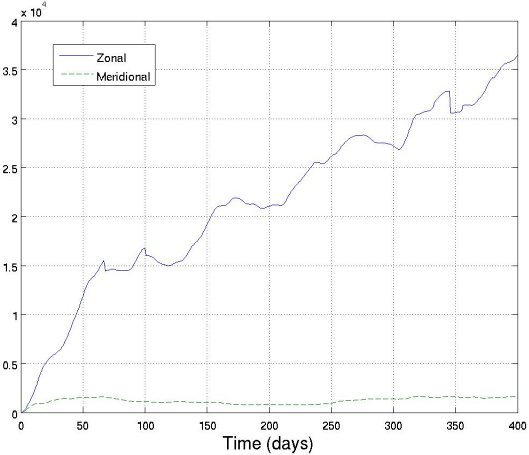
e.g. Zambianchi and Griffa, 1994

Mean flow effects



JHL et al., 2014

DIMES diffusivities



JHL et al., 2014

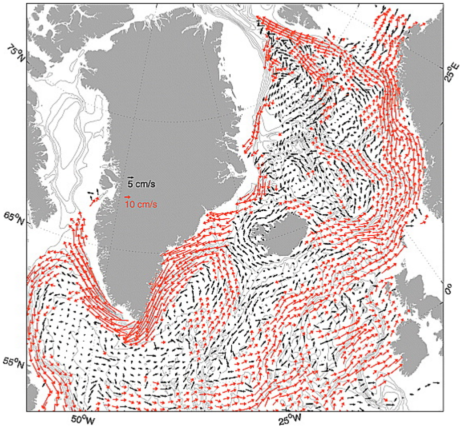
Practical solutions

R.E. Davis (1983, 85, 87, 91) proposed methods for extracting diffusivities in the presence of mean flows and inhomogeneities:

- Map the mean flow in geographical bins
- Subtract the mean velocities
- Calculate diffusivites from the *residual velocities*

A popular method of Lagrangian analysis

Mean flow



Jakobsen et al., 2003

Diffusivities

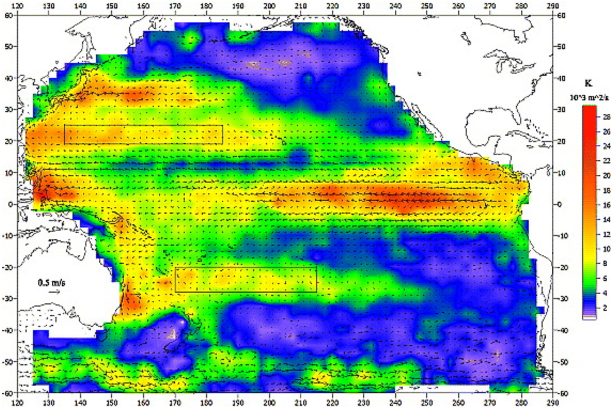
Various approaches used:

• Many calculate diffusivities from binned trajectories or extract equal length segments and average (e.g. Lumpkin et al., 2003)

• Zhurbas and Oh (2003, 2004) calculate a 2D diffusivity tensor, then extract the minor principal axis component

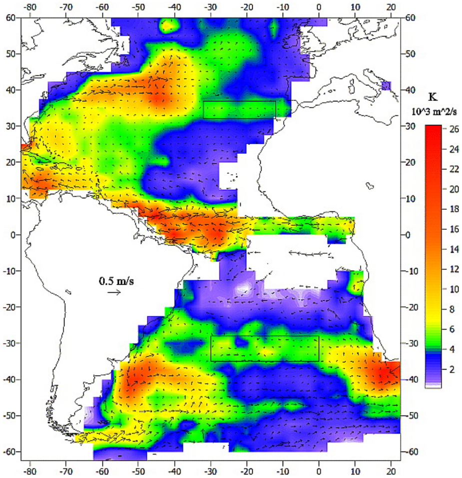
• Koszalka et al. (2011) use equal length segments and average using a clustering algorithm

Diffusivity maps



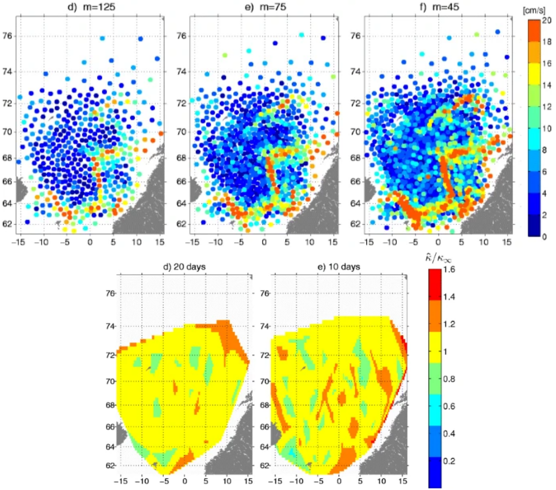
Zhurbas and Oh, 2004

Diffusivity maps



Zhurbas and Oh, 2004

Clustered maps



Koszalka and JHL, 201o

Summary: Single particle statistics

- Under general conditions, particles spread diffusively, as in a random walk
- The velocity autocorrelation determines the Lagrangian time scale and the diffusivity
- The diffusivity can be measured in (at least) 5 ways
- Mean flows have a profound effect on diffusivities parallel to the current

Summary: single particle statistics

Single particle dispersion has *generic limits*:

$$lim_{t \to 0}\ K = \nu^2 t$$

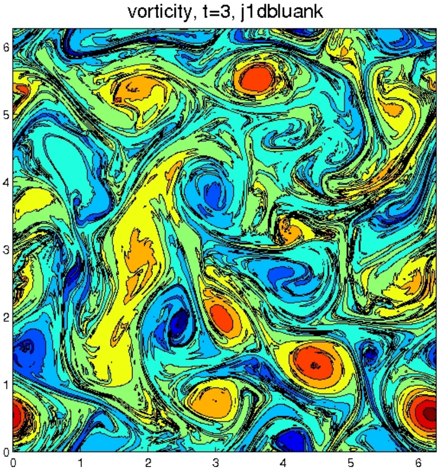
and

$$lim_{t \to \infty}\ K = \nu^2 T_L$$

So single particle dispersion is not really useful for distinguishing different flows (large scale turbulence, small scale, time dependence, etc.)

For this we turn to particle *pairs*

Exercises: 2-D turbulence model



2-D turbulence model

Solves the vorticity equation:

$$\frac{\partial}{\partial t}\zeta + J(\psi, \zeta) = \mathcal{F} + \mathcal{D}$$

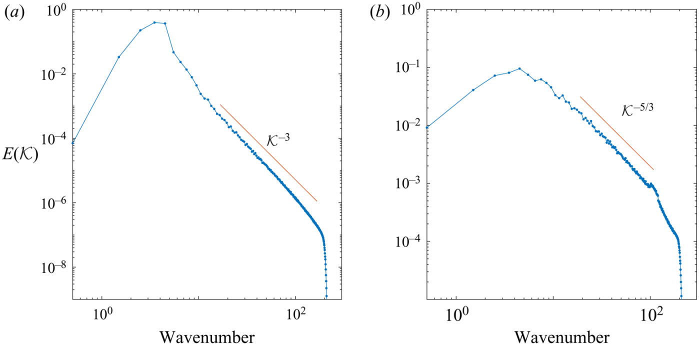
with:

$$\zeta = \nabla^2 \psi, \quad J(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial y}$$

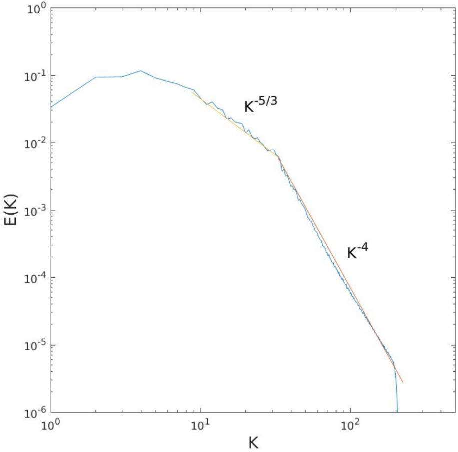
$\mathcal{F}$ is forcing, $\mathcal{D}$ the dissipation

Examine particle trajectories from 3 different flows

Spectra: cases 1 and 2

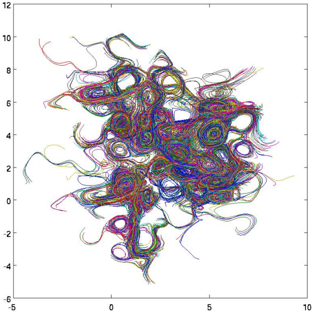


Spectrum: case 3



Particles

In each case, the model was run until the energy was statistically stationary, then 2000 particles were released



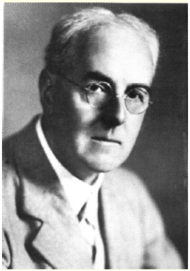
Tasks

- Load the data and plot the trajectories. Do you see differences between the three sets?
- Calculate the particle velocities. Plot time series for individual particles. Can you estimate a "memory" time?
- Calculate the single particle autocorrelation, $R(t)$, for a single particle. How quickly does this decay? Does this agree with your visual assessment?
- Average the autocorrelations for all the particles in a set. Plot the result. If the decay time is much less than the length of integration, you can consider breaking the time series into segments. This will produce better averages.

Tasks

- Calculate the Lagrangian time by integrating the autocorrelation in time. Do this in both x and y directions. Are they consistent?
- Calculate the diffusivities, $K_x(t)$ and $K_y(t)$. Do they asymptote to a constant value? Are they consistent with expectations from the trajectories?
- Plot the single particle dispersions, $\overline{(x-x_0)^2}$ and $\overline{(y-x_0)^2}$. How do these compare?
- Calculate the diffusivities by plotting the dispersions divided by 2t. How do these compare to the curves calculated from the autocorrelations?
- Now calculate the diffusivities for all three data sets. Can you distinguish the different sets?

Richardson (1926): Atmospheric diffusion shown on a distance-neighbour graph



$$\frac{\partial}{\partial t} C = \frac{\partial}{\partial y}(\kappa \frac{\partial}{\partial y} C)$$

Richardson (1926)

Diffusivity

Richardson found that the diffusivity wasn't constant but varied with the width of the plume:

$$\kappa \propto y^{4/3}$$

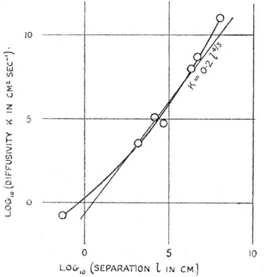
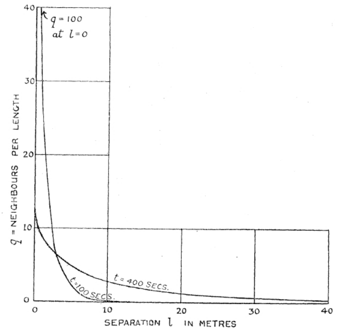


Fig. 8.

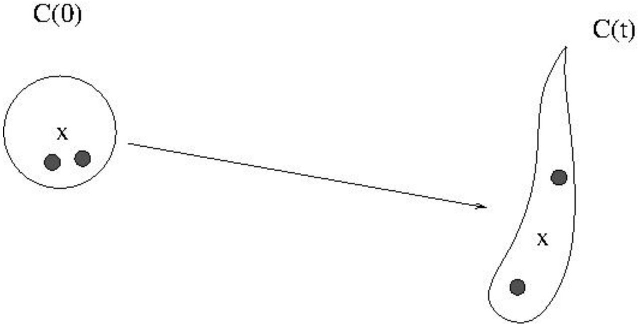
Tracer dispersion

Obtained a (self-similar) solution to the diffusion equation with the variable diffusivity:

$$\frac{\partial}{\partial t}C = \frac{\partial}{\partial y}(\beta y^{4/3}\frac{\partial C}{\partial y})$$



Tracer spreading

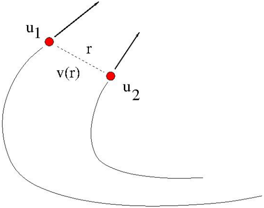


Pair dispersion

The cloud variance is equal to the mean square separation of all *pairs* of particles:

$$\overline{(\mathbf{r} - \bar{\mathbf{r}})^2} = \overline{(r_i - r_j)^2}$$

→ Measure tracer spreading using pairs of particles



Batchelor (1952), JHL (2008)

Two particle statistics

Relative dispersion:

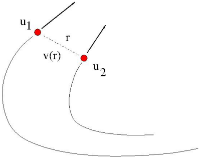
$$\overline{r^2} \equiv \frac{1}{N_{pairs}} \sum_{i \neq j} |\mathbf{x}_i - \mathbf{x}_j|^2$$

Relative diffusivity:

$$\begin{aligned} K_2 &= \frac{1}{2} \frac{d}{dt} \overline{r^2} \\ &= \overline{v(t)\, r(t)} \\ &= \overline{v(t)\, r(0)} + \int_0^t \overline{v(t) v(t')}\, dt' \end{aligned}$$

(e.g. Babiano et al., 1990)

Uncorrelated motion



Assuming a homogeneous flow:

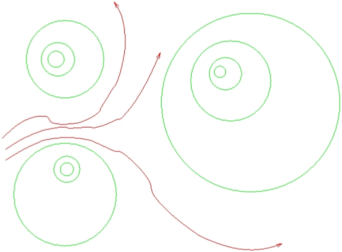
$$\begin{aligned}\overline{v(t)v(t')} &= \overline{(u_{1p}(t)-u_{2p}(t))(u_{1p}(t')-u_{2p}(t'))}\\ &= 2\overline{u_{ip}(t)u_{ip}(t')} - 2\overline{u_{1p}(t)u_{2p}(t')}\\ &= 2R(t') - 2\overline{u_{1p}(t)u_{2p}(t')}\end{aligned}$$

So:

$$K_2 = 2K_1 - 2\int_0^t \overline{u_{1p}(t)u_{2p}(t')}\,dt$$

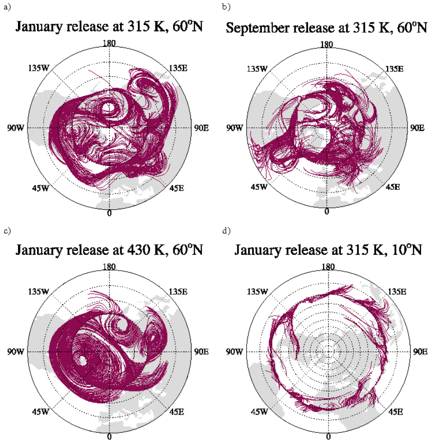
Uncorrelated motion

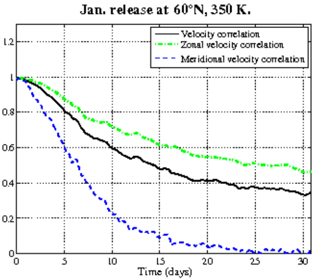
At large separations, the pair velocities are *uncorrelated*



So $lim_{t \to \infty} K_2 = 2K_1$

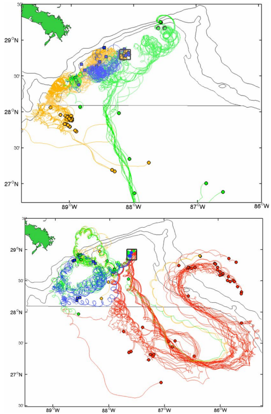
Velocity correlation: atmosphere



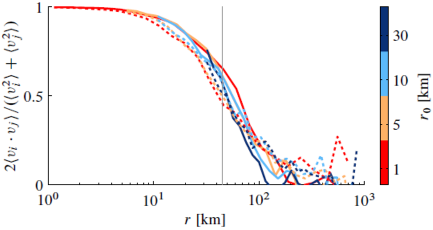


Graff et al. (2015)

Velocity correlation: Gulf of Mexico



Poje et al. (2014)



Beron-Vera and JHL (2016)

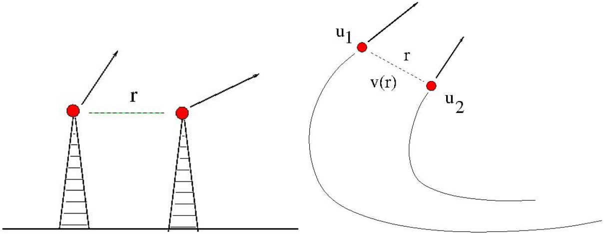
Relative dispersion

- When pair motion is decorrelated, *relative* (two particle) dispersion becomes *absolute* (single particle) dispersion
- When pair motion is correlated, relative dispersion depends on the energetic characteristics of the flow
- Provides *a link between Eulerian and Lagrangian statistics*

Structure functions

For a homogeneous, isotropic flow:

$$\delta v_L(r) = \delta v_E(r)$$



$$S_n(r) \equiv \overline{(|u_{1p} - u_{2p}|^n)_r} = \overline{v(r)^n}$$

Structure functions

The second order structure function ($n = 2$) is related to the energy spectrum:

$$\begin{aligned}\overline{v(r)^2} &= \overline{(u_{1p}(x+r,t) - u_{2p}(x,t))^2} \\ &= 2\int_0^\infty E(k)[1 - \frac{2J_1(kr)}{kr}]\,dk\end{aligned}$$

where $E(k)$ is the energy spectrum and J_1 is a Bessel function (Bennett, 1984; JHL, 2016).

Local dispersion

Assume a power law spectrum:

$$E(k) \propto k^{-\alpha}$$

Can show the integral diverges if $\alpha \geq 3$ (steep spectra) or if $\alpha \leq 1$ (shallow spectra). For intermediate values ($1 < \alpha < 3$):

$$\overline{v(r)^2} \propto r^{\alpha-1} \; .$$

and:

$$K_2 = \frac{1}{2}\frac{d}{dt}\overline{r^2} \propto r^{(\alpha+1)/2}$$

Non-local dispersion

If the spectrum is steep ($\alpha \geq 3$):

$$\overline{v(r)^2} \propto r^2$$

Also:

$$K_2 \propto r^2$$

Kolmogorov (1941) energy spectrum

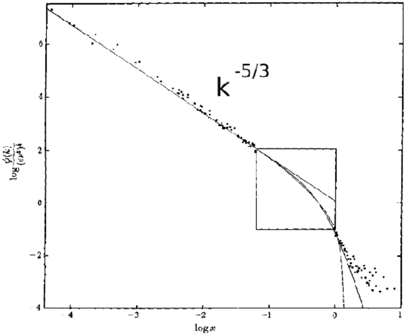


FIGURE 12. Seventeen spectra compared to the theories of Kolmogoroff, Heisenberg and Kovasznay. The straight line has a slope of $-\frac{5}{3}$, the curved solid line is Heisenberg's theory and the dashed line is Kovasznay's theory. Within the square, the observations are too crowded to display on this scale and they are shown in figure 13.

Grant et al. (1962)

Richardson dispersion

$$S_2 = \overline{v^2} \propto r^{2/3}$$

Kolmogorov's "2/3 Law". Similarly, the diffusivity is:

$$K_2 \propto r^{4/3}$$

→ Consistent with Richardson's (1926) observations. So smoke dispersion probably due to 3D turbulence.

Obukhov (1941), Batchelor (1952)

Diffusivity

But Richardson found that the diffusivity wasn't constant, but varied with the width of the plume:

$$\kappa \propto y^{4/3}$$

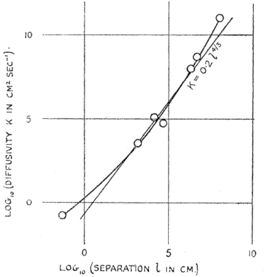


Fig. 8.

Probability density functions

Richardson (1926) proposed:

$$\frac{\partial}{\partial t} C = \frac{\partial}{\partial r}(\beta r^{4/3} \frac{\partial}{\partial r} C)$$

Similarly, the **PDF of pair separations**, $p(r, t)$, obeys:

$$\frac{\partial}{\partial t} p = \frac{\partial}{\partial r}(\beta r^{4/3} \frac{\partial}{\partial r} p)$$

- Can solve the equation for this and other K_2
- Then can determine all the moments

Bennett (2006), JHL (2010)

Uncorrelated motion

$$K_2 = const.$$

Asymptotic solution:

$$p(r,t) = \frac{1}{4\pi K_2 t} exp(-\frac{r^2}{4K_2 t})$$

Dispersion: $\overline{r^2} = 4K_2 t$ Kurtosis: $Ku = \frac{\overline{r^4}}{(\overline{r^2})^2} = 2$

Turbulent energy cascade

$$E(\kappa) \sim \kappa^{-5/3}, \quad \kappa = \beta r^{4/3}$$

Asymptotic solution:

$$p(r,t) = (\frac{3}{2})^5 \frac{1}{4\pi\beta^3 t^3}\, exp(-\frac{9r^{2/3}}{4\beta t})$$

Dispersion: $\overline{r^2} = 5.2675\, \beta^3\, t^3$ Kurtosis: $Ku = 5.6$

Non-local spectrum

$$E(\kappa) \sim \kappa^{-3} \text{ or steeper}, \quad \kappa = r^2/T$$

Solution:

$$p(r,t) = \frac{1}{4\pi(\pi t/T)^{1/2}\, r_0^2}\, exp(-\frac{[ln(r/r_0) + 2t/T]^2}{4t/T})$$

Dispersion: $\overline{r^2} = r_0^2 exp(\frac{8t}{T})$ Kurtosis: $Ku = exp(\frac{8t}{T})$

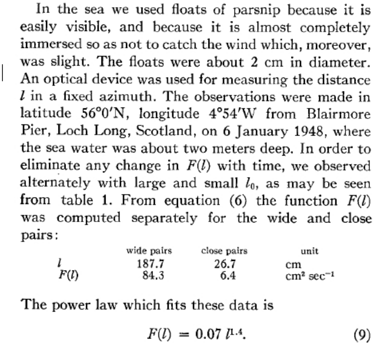
Comments

- Relative dispersion applies to error growth—if the particle is moved a small distance from its initial position, it will diverge. Relative dispersion *useful for search and rescue operations*.

- Exponential growth implies a *sensitive dependence on initial conditions*. This is Lagrangian **chaos**

- Relative dispersion is exponential at sub-grid scales in models, because the energy spectrum is artifically steep

Loch Long, Scotland



In the sea we used floats of parsnip because it is easily visible, and because it is almost completely immersed so as not to catch the wind which, moreover, was slight. The floats were about 2 cm in diameter. An optical device was used for measuring the distance l in a fixed azimuth. The observations were made in latitude 56°0′N, longitude 4°54′W from Blairmore Pier, Loch Long, Scotland, on 6 January 1948, where the sea water was about two meters deep. In order to eliminate any change in $F(l)$ with time, we observed alternately with large and small l_0, as may be seen from table 1. From equation (6) the function $F(l)$ was computed separately for the wide and close pairs:

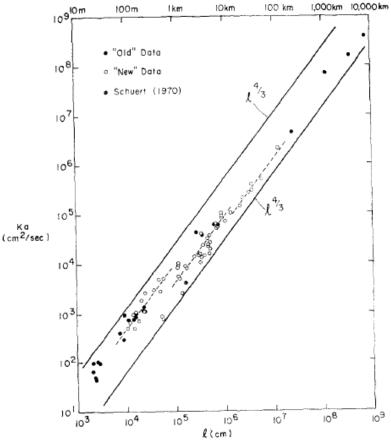
	wide pairs	close pairs	unit
l	187.7	26.7	cm
$F(l)$	84.3	6.4	cm² sec⁻¹

The power law which fits these data is

$$F(l) = 0.07\, l^{1.4}. \tag{9}$$

Richardson and Stommel (1948)

Surface dispersion

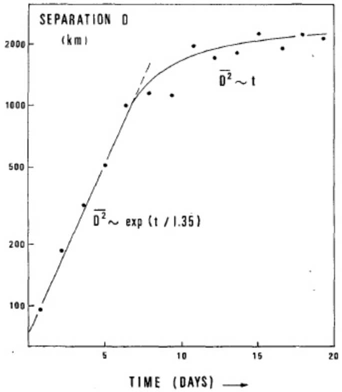


Okubo (1970)

EOLE

483 balloons in the Southern Hemisphere stratosphere

EOLE balloons



Morel and Larcheveque (1974)

TWERLE balloons

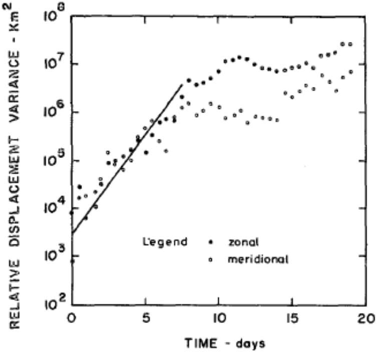
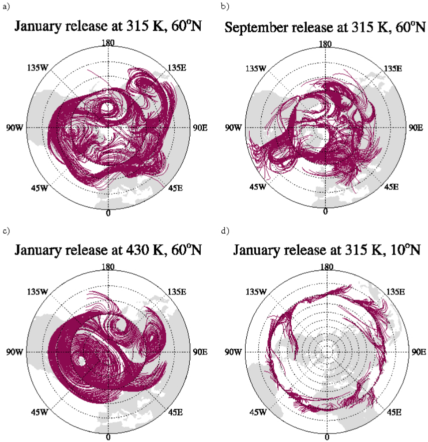


FIG. 6. The mean-square relative displacement components for midlatitudes releases on a log-linear scale. The straight line indicates an exponential region.

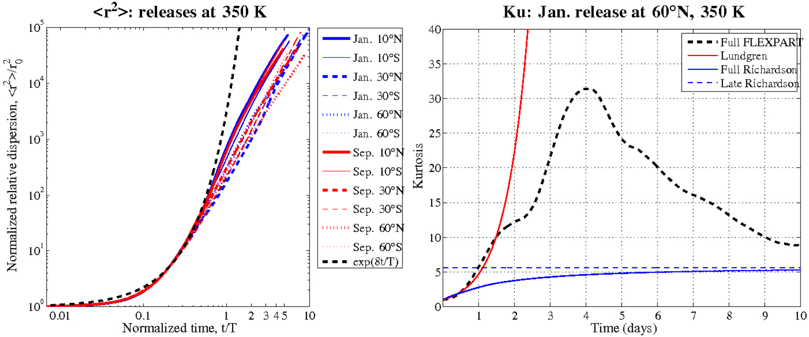
Er-el and Peskin (1981)

Synthetic particles in the atmosphere



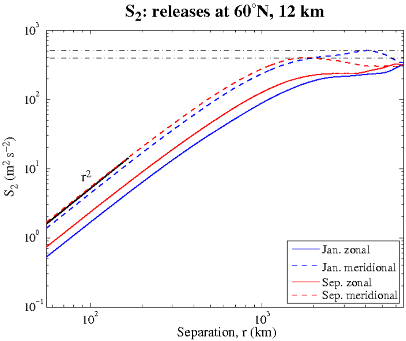
Graff et al. (2015)

Synthetic particles in the atmosphere



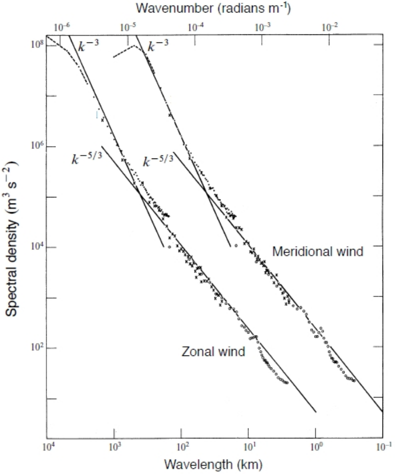
Graff et al. (2015)

Synthetic particles



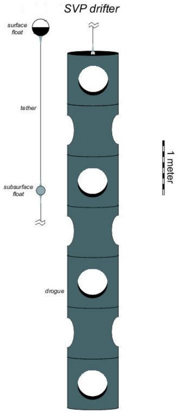
Graff et al. (2015)

Atmospheric spectra

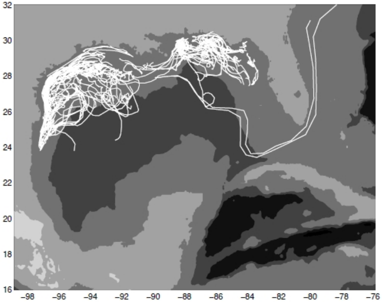


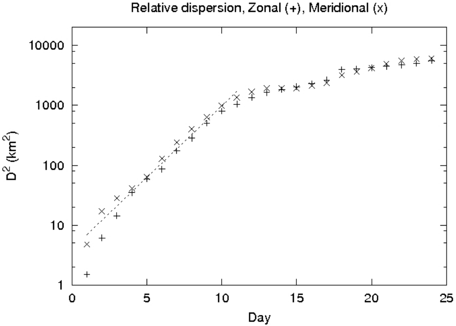
Nastrom and Gage (1985)

Surface drifters



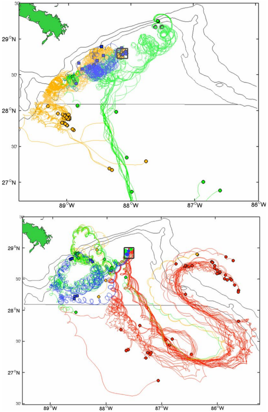
SCULP drifters: Gulf of Mexico



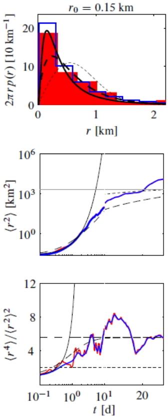


JHL and Ohlmann (2003)

GLAD experiment in the Gulf of Mexico

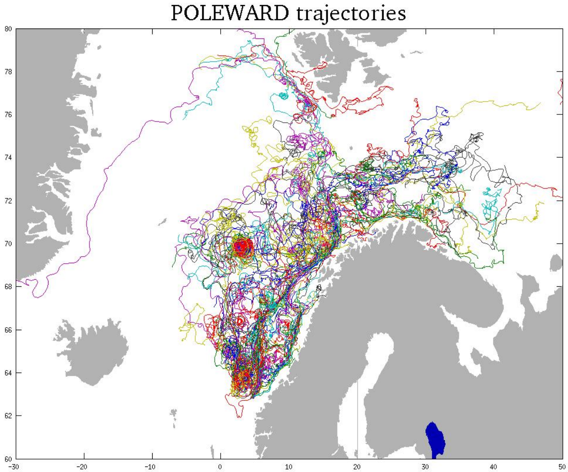


Poje et al. (2014)



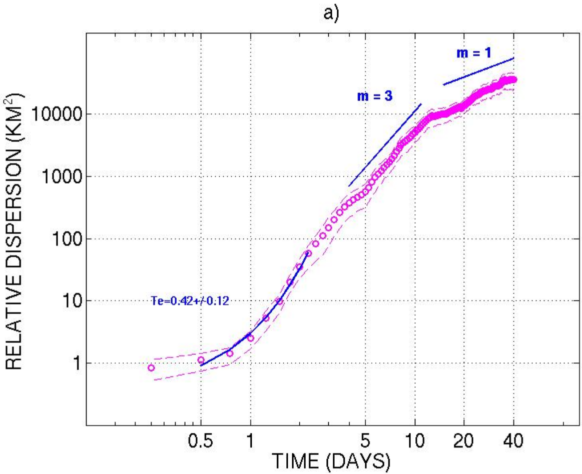
Beron-Vera and JHL (2016)

POLEWARD drifters: Nordic Seas



Koszalka et al. (2009)

POLEWARD drifters: Nordic Seas

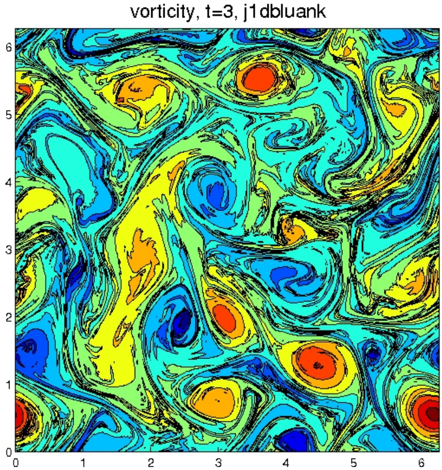


Koszalka et al. (2009)

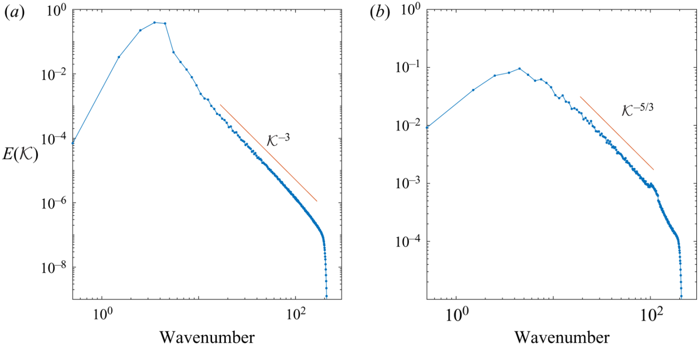
Summary: pair statistics

- Pair dispersion reflects the spreading of a tracer cloud
- When pair velocities are correlated, the dispersion depends on the energy spectrum
- The structure functions, diffusivity and PDFs can be deduced for power law spectra
- Distinguish “local” and “non-local” spectra
- Observations suggest *exponential growth* below the deformation radius in the lower stratosphere and at the ocean surface
- Implies the sub-deformation scale energy spectrum is at least as steep as κ^{-3}
- Thus Lagrangian statistics can be used to infer Eulerian spectra

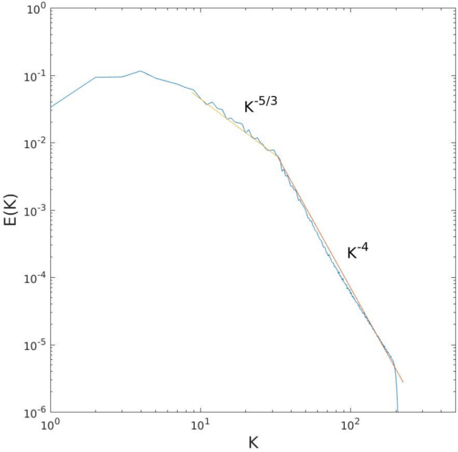
2-D turbulence model



Spectra: cases 1 and 2



Spectrum: case 3



Tasks

- Using the code provided, isolate all pairs which are one grid cell away from each other ($r(0) \leq 0.01$) from one data set
- Calculate the correlation of the separation velocites. Over what times and separations is the motion correlated?
- Calculate the relative dispersion. Does it look exponential, power law, both or neither?
- Calculate the separation kurtosis. How does this compare to theory.
- Now calculate the relative dispersion and kurtosis for the other data sets. Can you match the spectra with the statistics?

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THANKS!

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