Fourier transform and its possible applications

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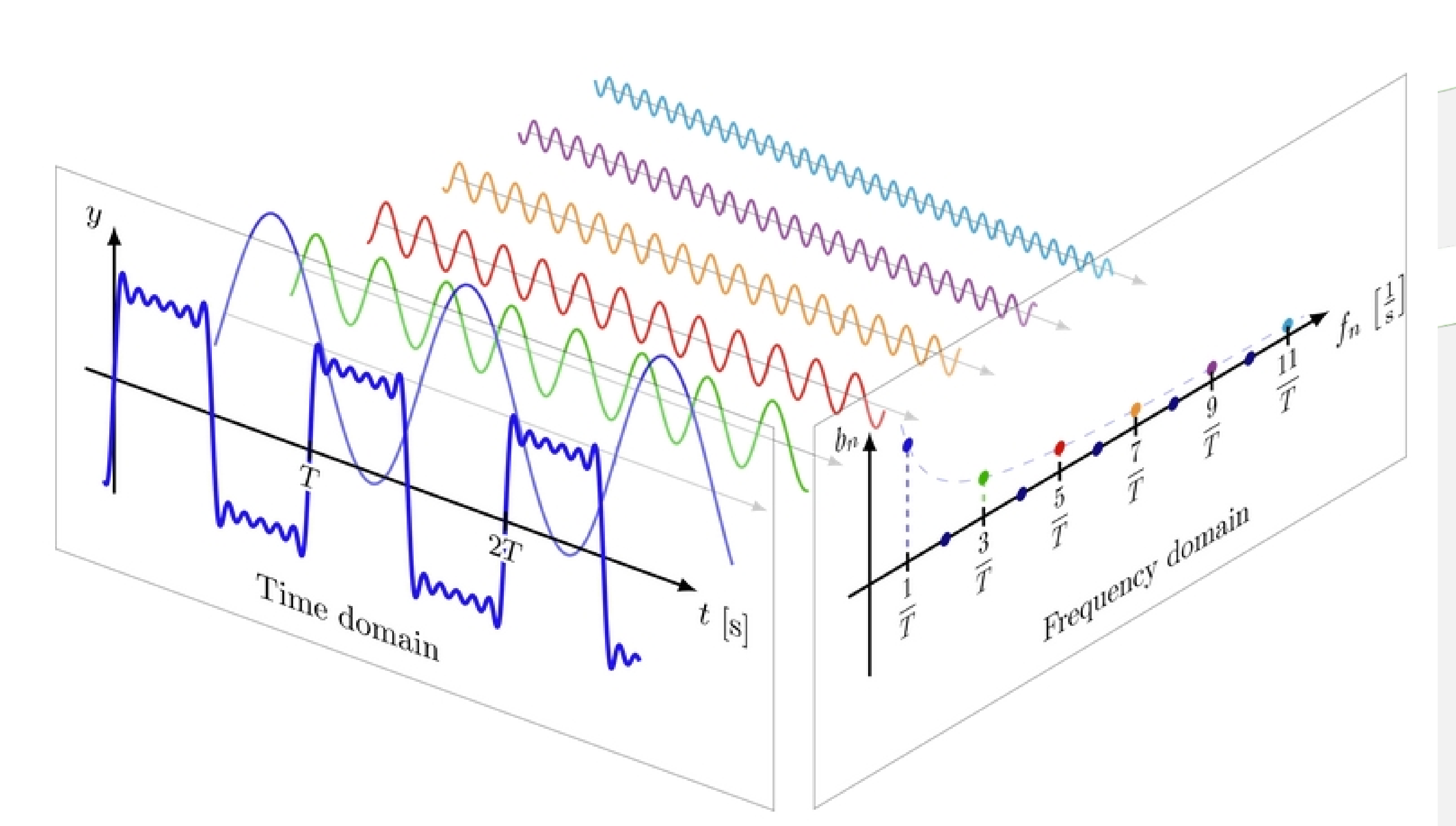
Fourier Series and Fourier Transforms

Objective:

Decompose a function f(t) (that satisfies certain conditions) into a sum of an *infinite number of sinusoidal terms*



Fourier series- Hypothesis

1) f(t) must be a periodic function: f(t) = f(t+nT), where T is the period

2) f(t) must be a continuous function (at least piecewise continuous), with a finite number of discontinuities of finite entity

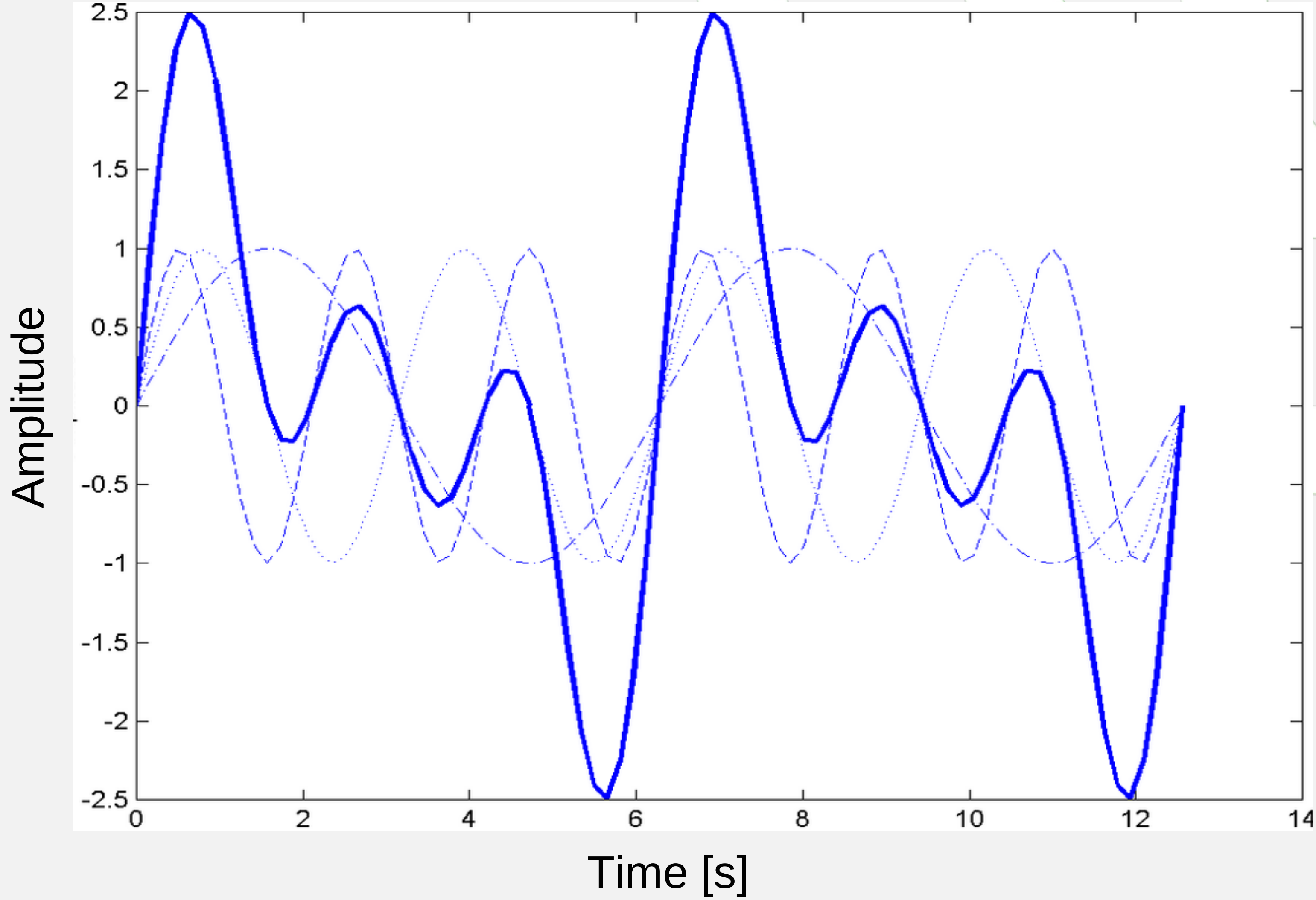
3) f(t) must contain a finite number of maxima and minima

4) The integral $\int_{-\infty}^{+\infty} |f(t)| dt$ must have a finite value

Because these are sufficient conditions, if at least one of them is verified the decomposition is applicable.
Practically the decomposition is applicable to all the known physical functions.

Example

f(t) periodic function composed by three sinusoids



Fourier Series

If f(t) is a periodic function: f(t)=f(t+nT)

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos(\omega_n t)+b_n \sin(\omega_n t)$$

with: $\omega_n=\frac{2\pi n}{T}$ *angular frequency*

$$a_n=\frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(\omega_n t)\,dt \quad \Rightarrow \quad a_0=\frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$$b_n=\frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(\omega_n t)\,dt$$

Some comments

if f(t) is an integrable function $a_n \; e \; b_n \Rightarrow 0 \quad for \quad n \Rightarrow \infty$

Given a function f(t): $f(t) \Leftrightarrow a_n \; e \; b_n$

One period of f(t) is sufficient: $(-T/2, T/2), \; (0, T), \; (-T, 0) \ldots$

ω depends only from the period T: $\omega_n = \frac{2\pi n}{T}$

ω_1 is the first harmonic $\omega_1 = \frac{2\pi}{T}$

By means of the Fourier series decomposition a function f(t) can be transformed from the time domain to the frequency domain...

f(t) as sum of (co)sinusoids

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty} C_n \cos(\omega_n t+\phi_n) \qquad (*)$$

$$C_n = \sqrt{a_n^2+b_n^2}$$ Amplitude spectrum (**)

$$\phi_n = \arctan\left(-\frac{b_n}{a_n}\right)$$ Phase spectrum (***)

(*) demonstrates that the Fourier series decomposition of the function f(t) corresponds to sum an infinite co-sinusoidal terms (called harmonics) to its mean value $a_0/2$. These co-sinusoids are at frequencies multiple of the fundamental frequency 1/T and have appropriate amplitude C_n and phase φ_n

Equations (**) and (***) give the amplitude and phase of each harmonic, i.e. of each frequency that composes the function f(t)

The ensemble of this information gives respectively the amplitude spectrum and the phase spectrum of the function f(t) which define unambiguously f(t)

Fourier series in exponential form

From Euler's formula:

$$\cos(\omega_n t) = (e^{j\omega_n t} + e^{-j\omega_n t})/2 \qquad \sin(\omega_n t) = -j(e^{j\omega_n t} - e^{-j\omega_n t})/2$$

the Fourier series can be written as:

$$f(t) = \sum_{-\infty}^{\infty} F_n e^{j\omega_n t}$$

with

$$F_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-j\omega_n t} dt$$

Introducing the negative frequencies we can simplify the formulation of the series decomposition

Fourier Transform

Now consider non periodic function, or periodic function where the period T is extended towards infinity

$$T \rightarrow \infty$$

$$\omega_n = \frac{2\pi n}{T} \Rightarrow$$ becomes the continuous variable ω

$$F_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-j\omega_n t} dt \Rightarrow F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Direct Fourier Transform

$$f(t) = \sum_{-\infty}^{\infty} F_n e^{j\omega_n t} \Rightarrow f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

Inverse Fourier Transform

- *The Fourier transforms allow us to change domain, from the time domain to the frequency domain, as the Fourier series*
- *They differ from the series in that the harmonic frequency ω, that in the series can only assume discrete values ω_n (the first harmonic and its multiples), now varies with continuity*

Fourier Transform:
Amplitude and Phase spectrum

F(ω) in general will be a complex number:

$$F(\omega) = a(\omega) - j\,b(\omega)$$

We can write

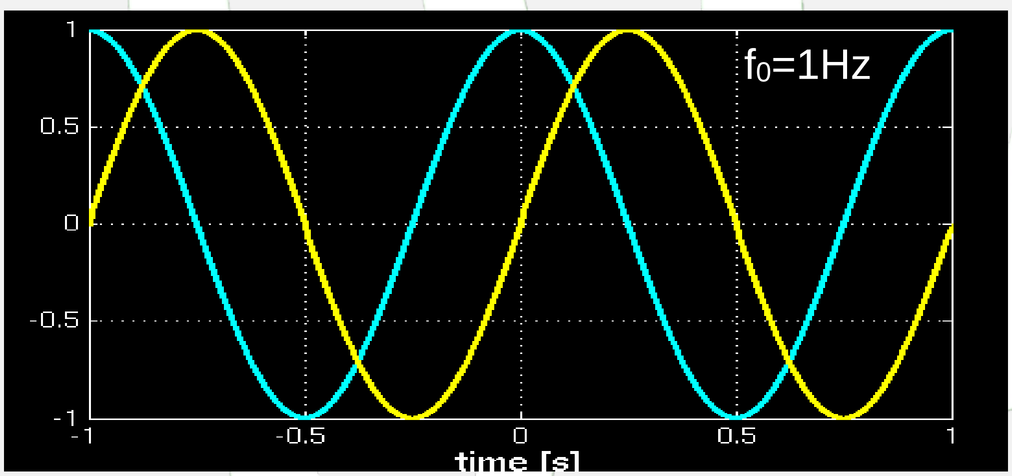
$$F(\omega) = |F(\omega)|\,e^{j\phi(\omega)}$$

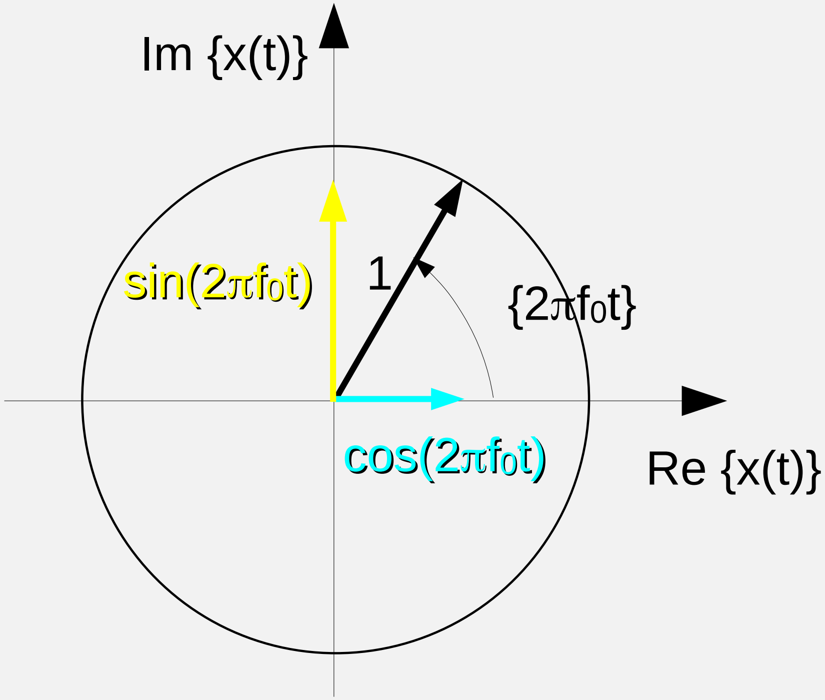
with $|F(\omega)| = \sqrt{a^2(\omega) + b^2(\omega)}$ Amplitude spectrum

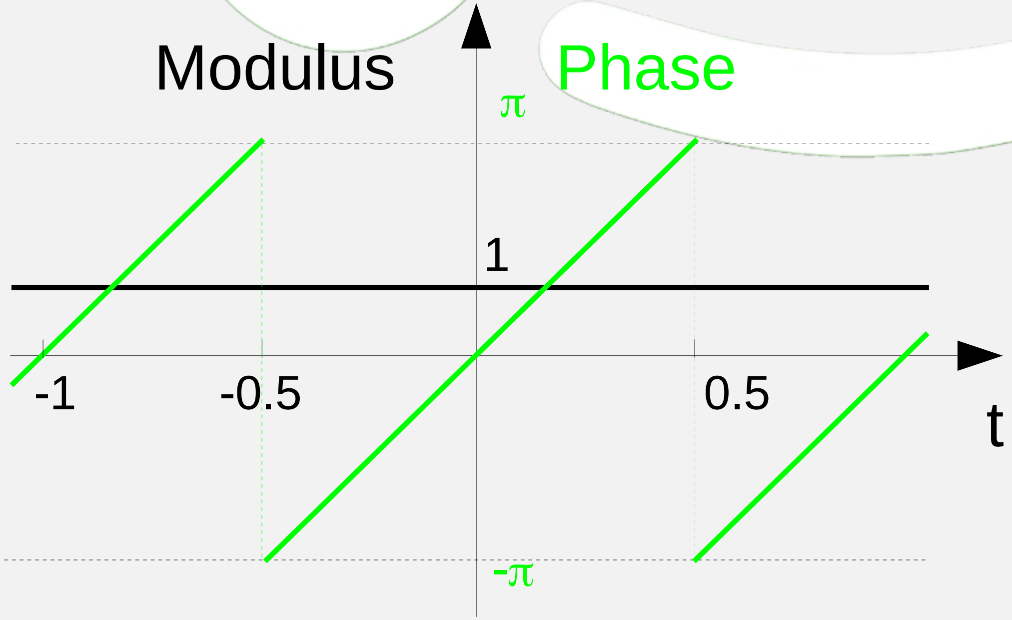
and $\phi(\omega) = \arctan\left(-\frac{b(\omega)}{a(\omega)}\right)$ Phase spectrum

Complex Exponential

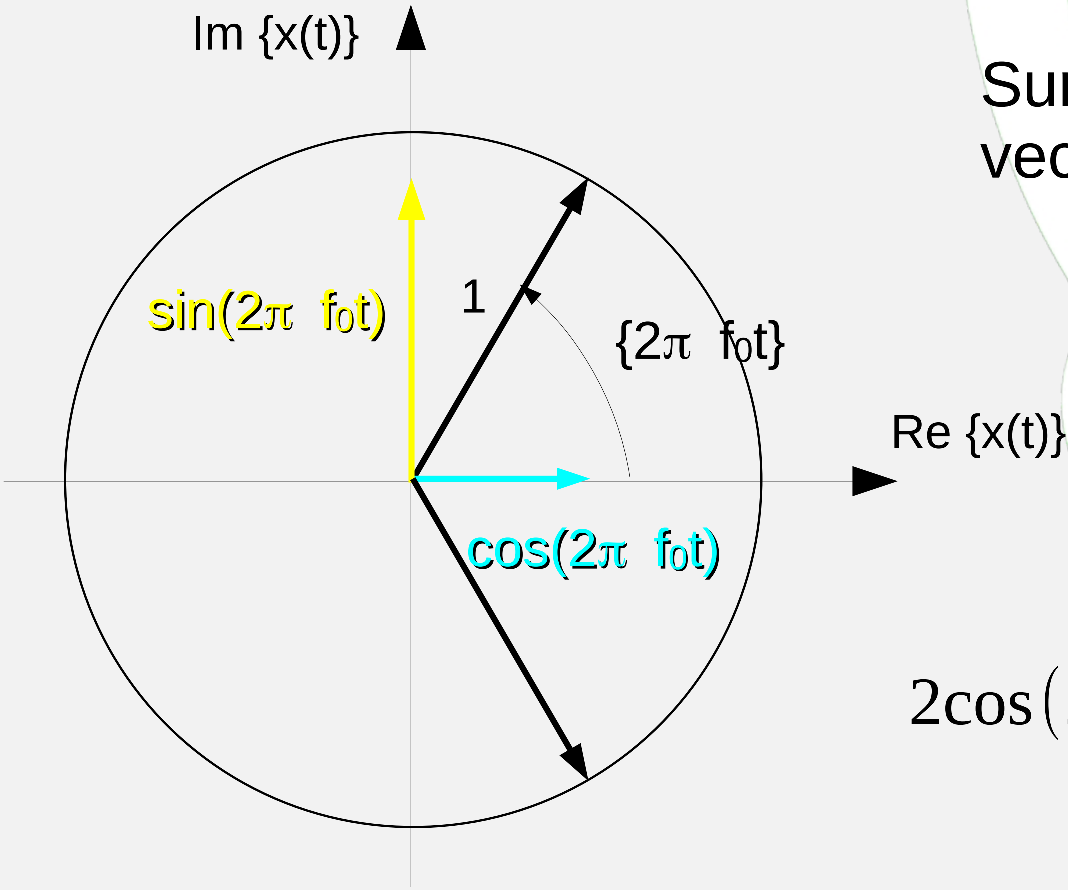
$$x(t) = \exp(j2\pi f_0 t)$$







Euler's Formula

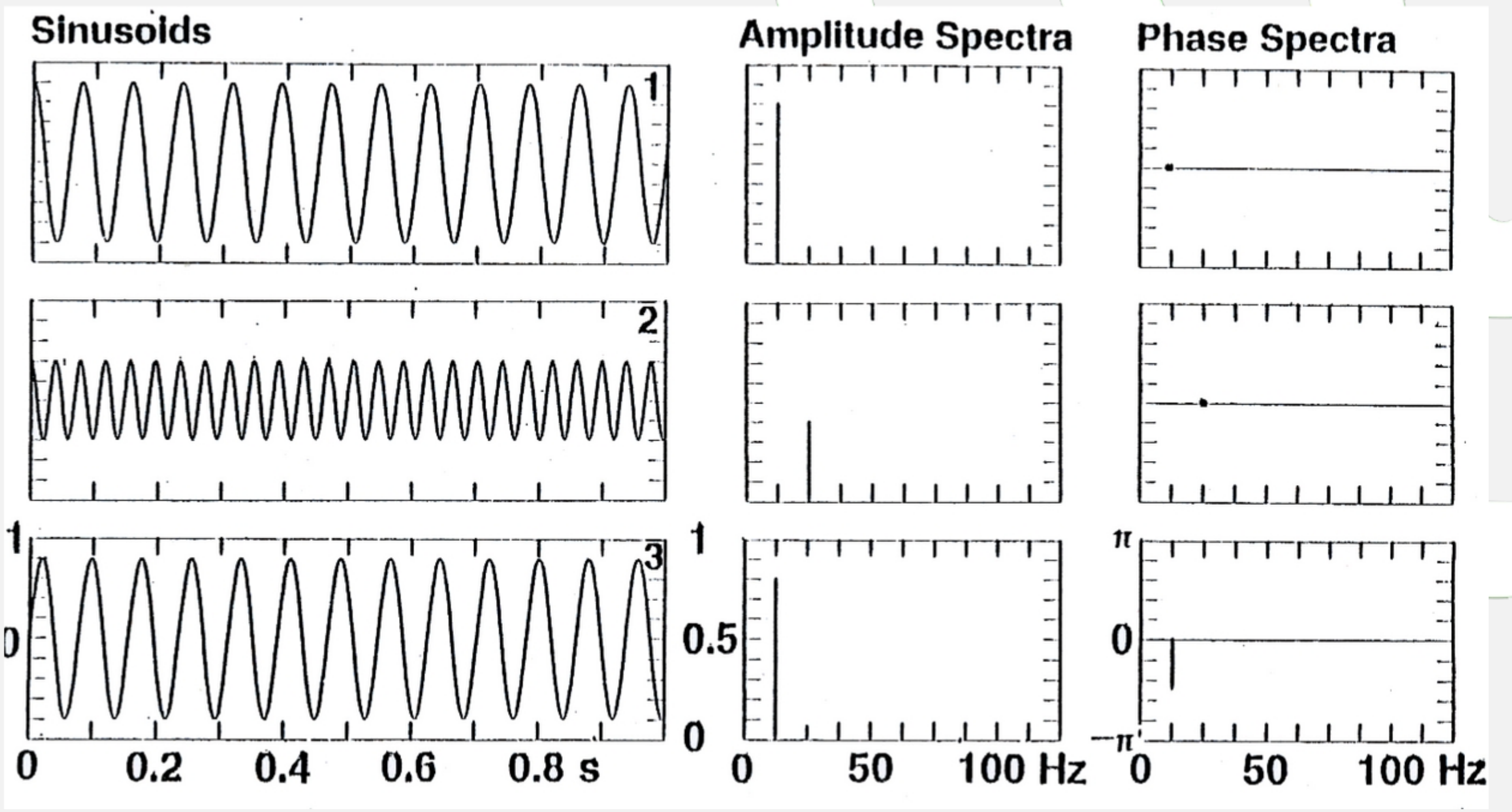


Sum and subtraction of vectors (parallelogram law)

$$2\cos(2\pi f_0 t) = \exp(j2\pi f_0 t) + \exp(-j2\pi f_0 t)$$

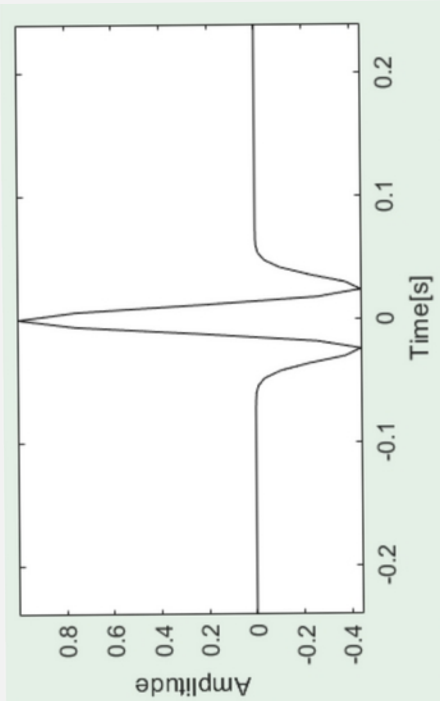
$$2j\,\sin(2\pi f_0 t) = \exp(j2\pi f_0 t) - \exp(-j2\pi f_0 t)$$

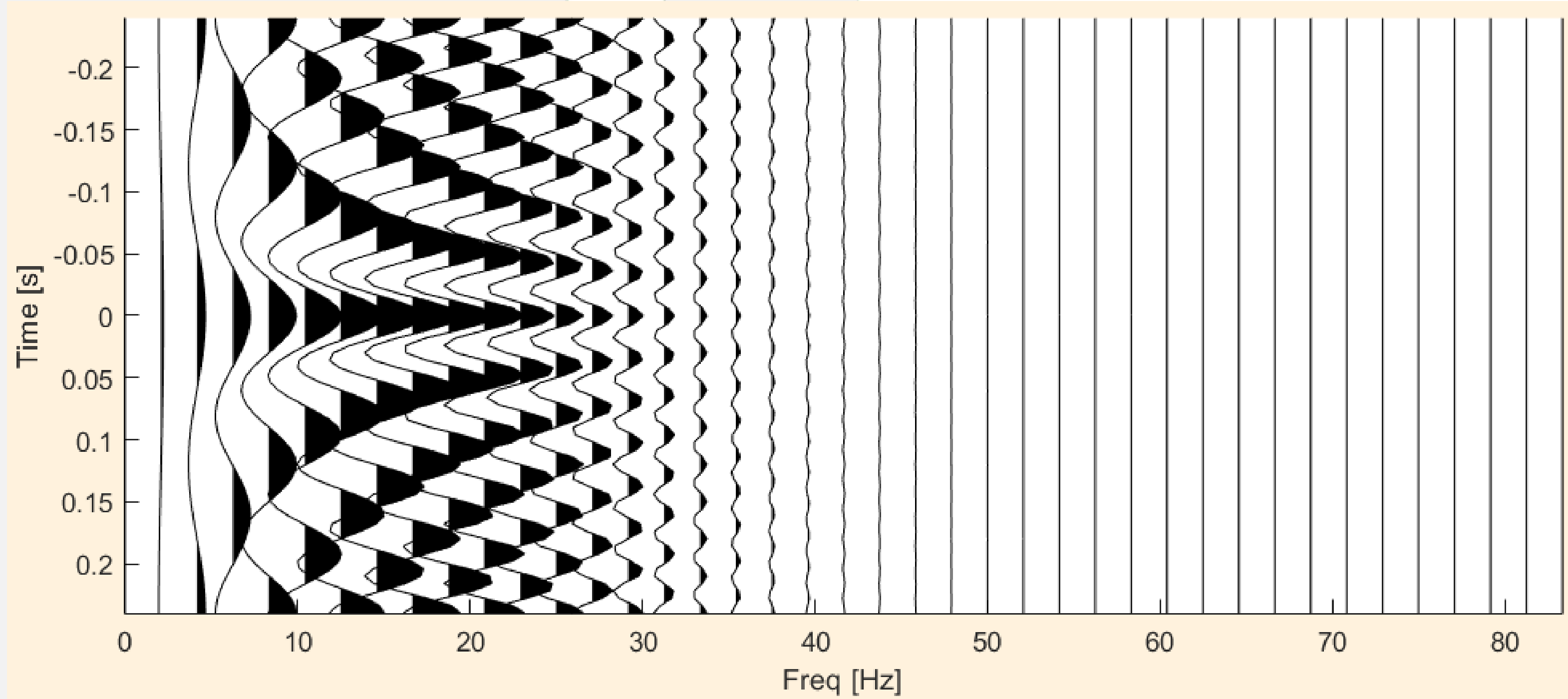
Examples of (co)sinusoids with the relative amplitude and phase spectra

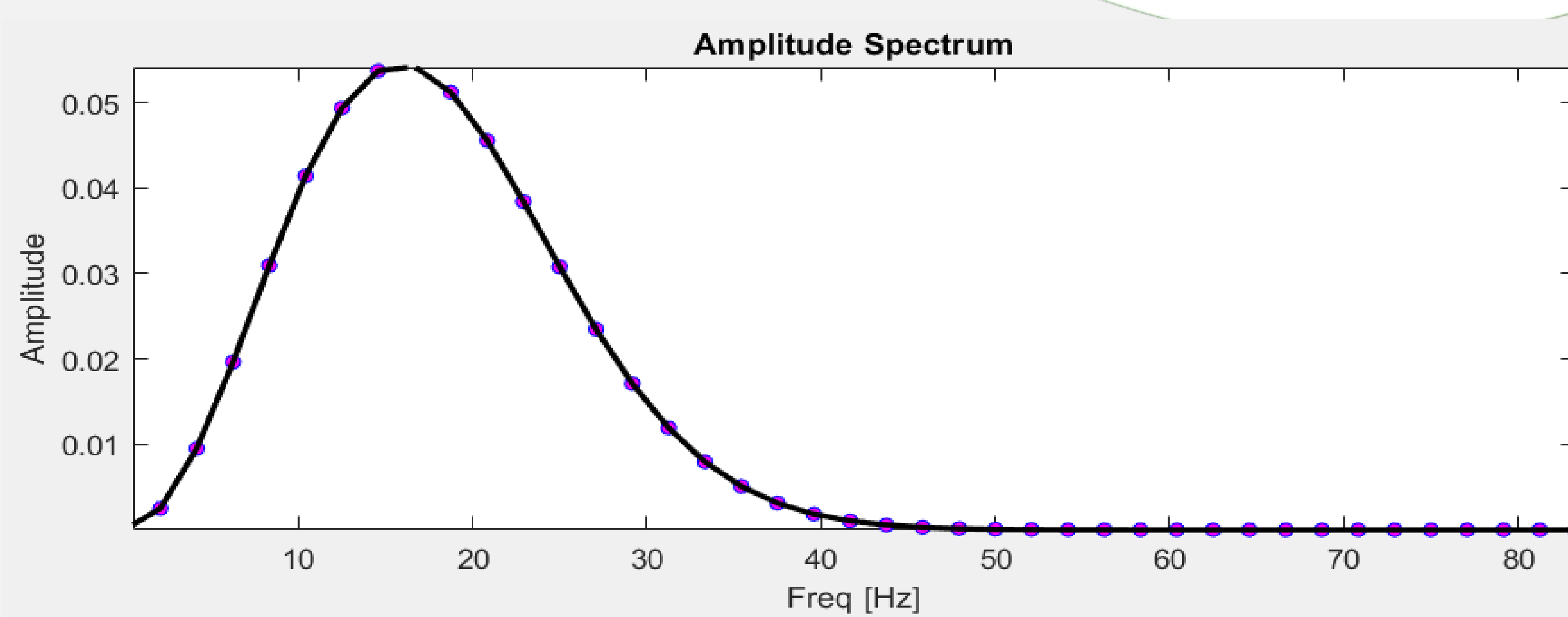


1) Co-sinusoid at	12.5 Hz	zero phase
2) Co-sinusoid at	25 Hz	zero phase
3) Co-sinusoid at	12.5 Hz	-90° phase

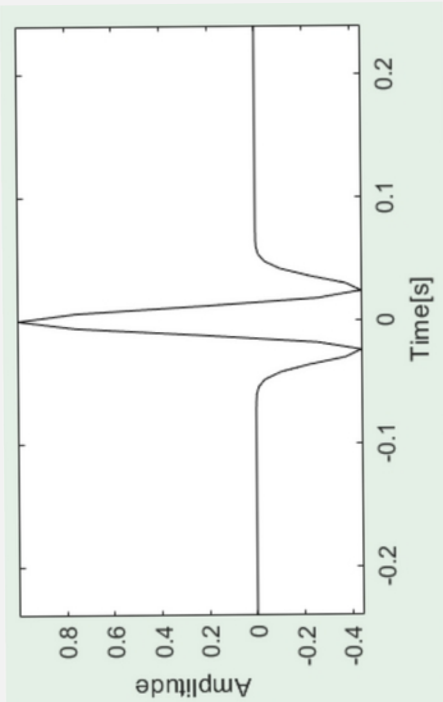
Example: Ricker Wavelet

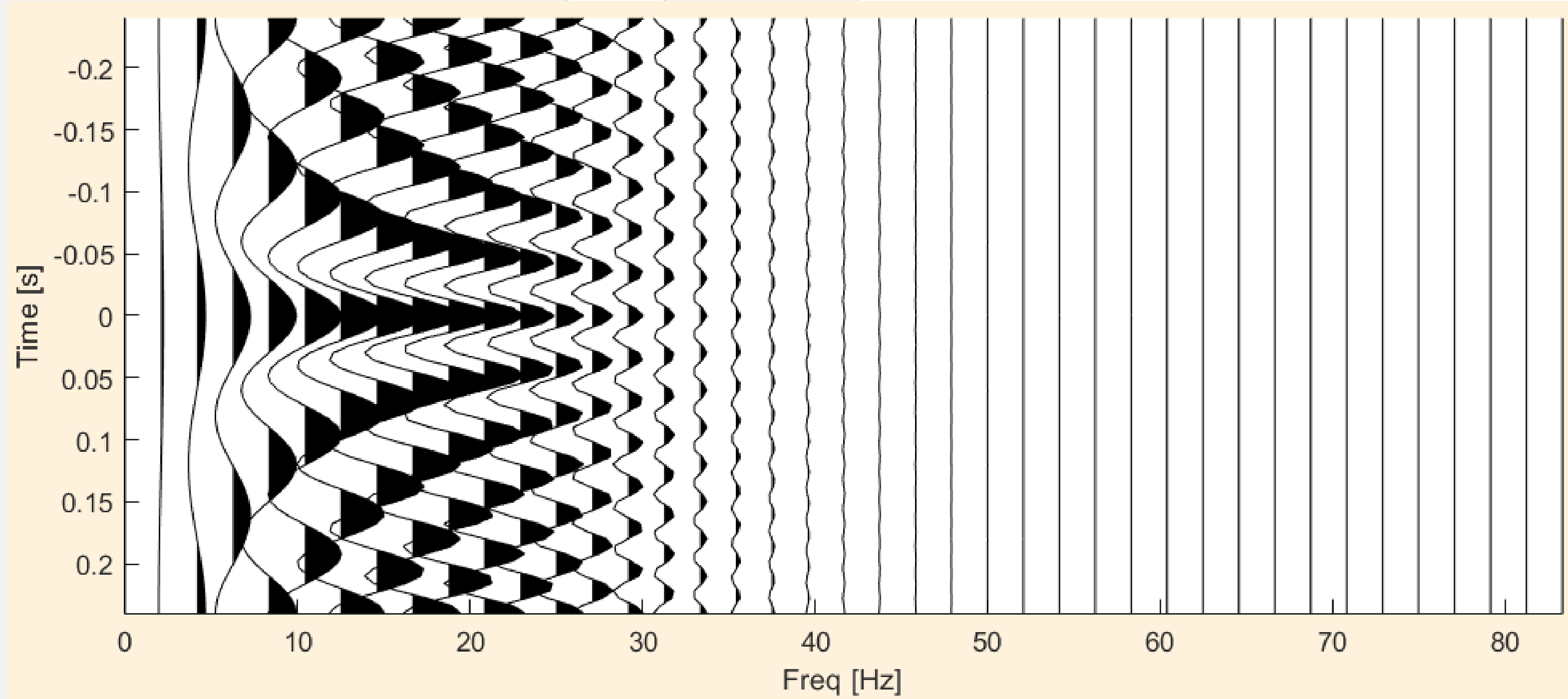


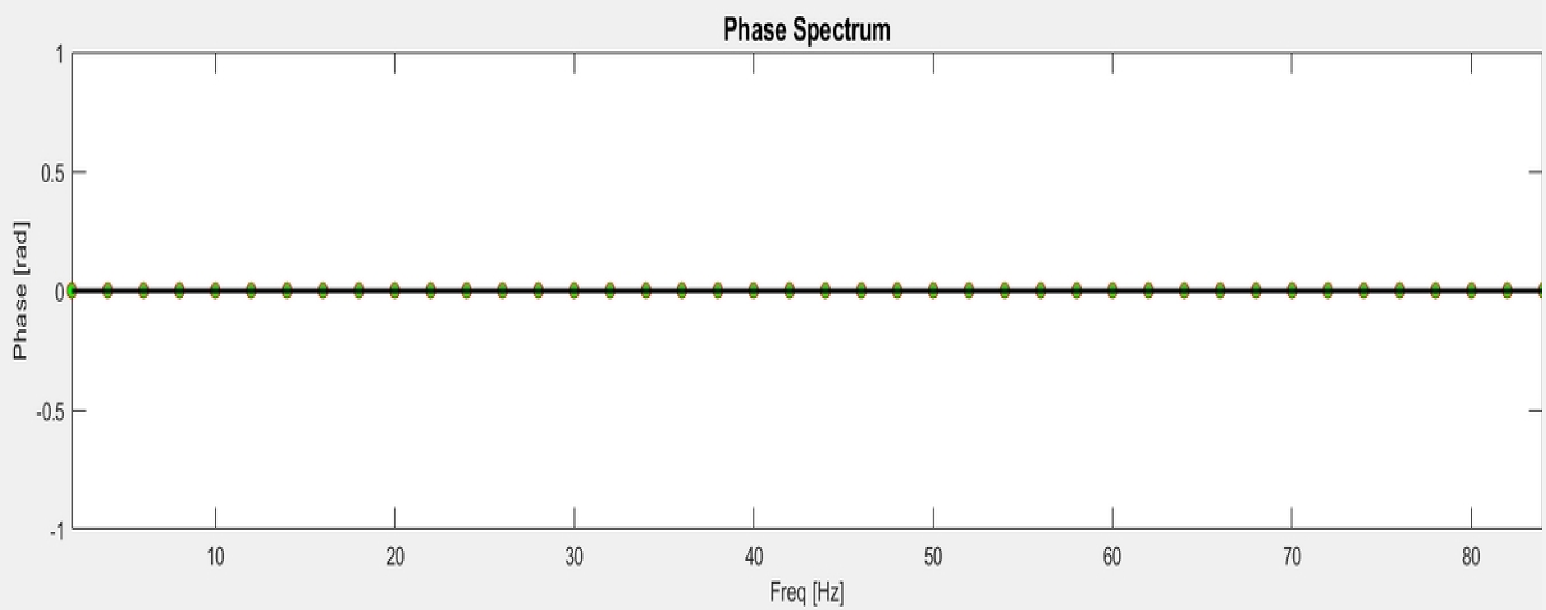




Example: Ricker Wavelet

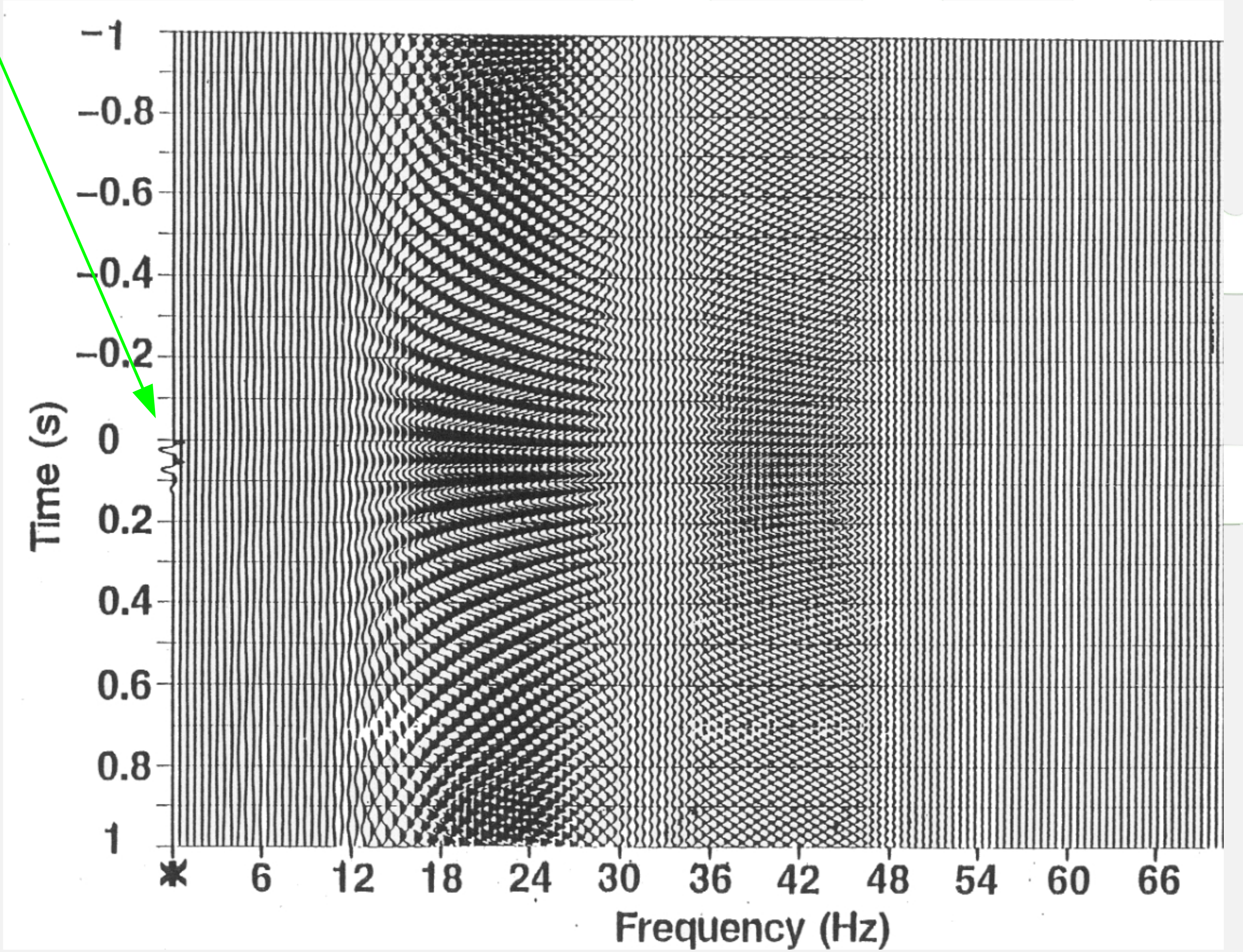






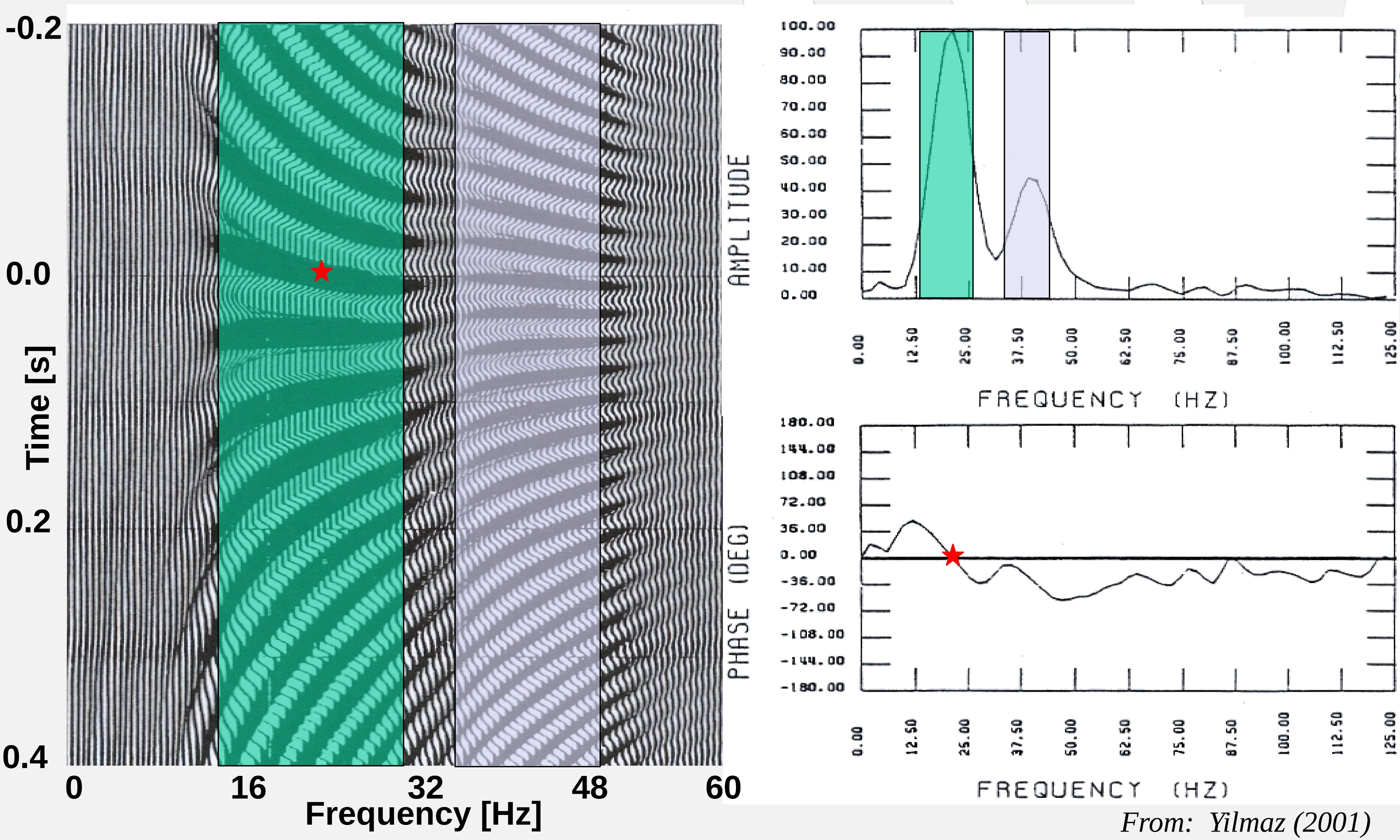
Example: generic wavelet

f(t) = seismic wavelet



Example: generic wavelet

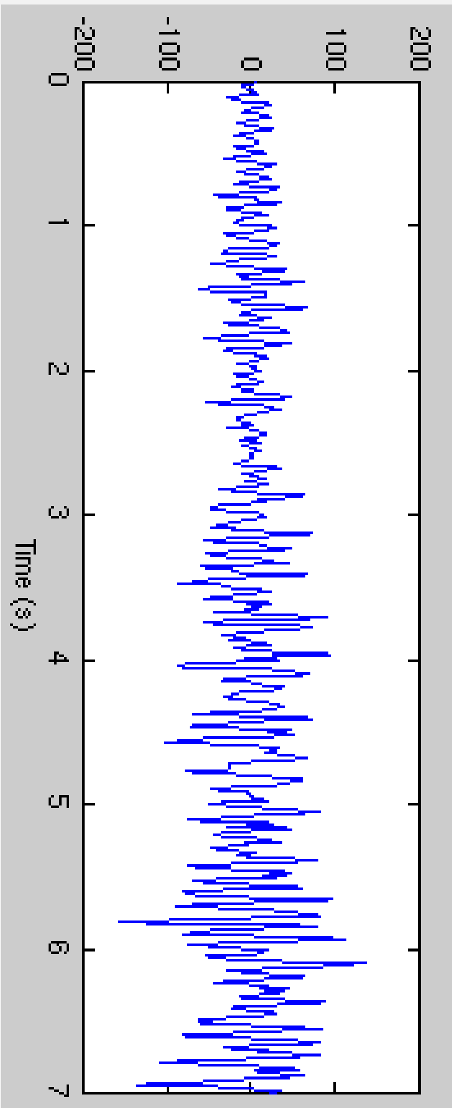
Amplitude and Phase spectra

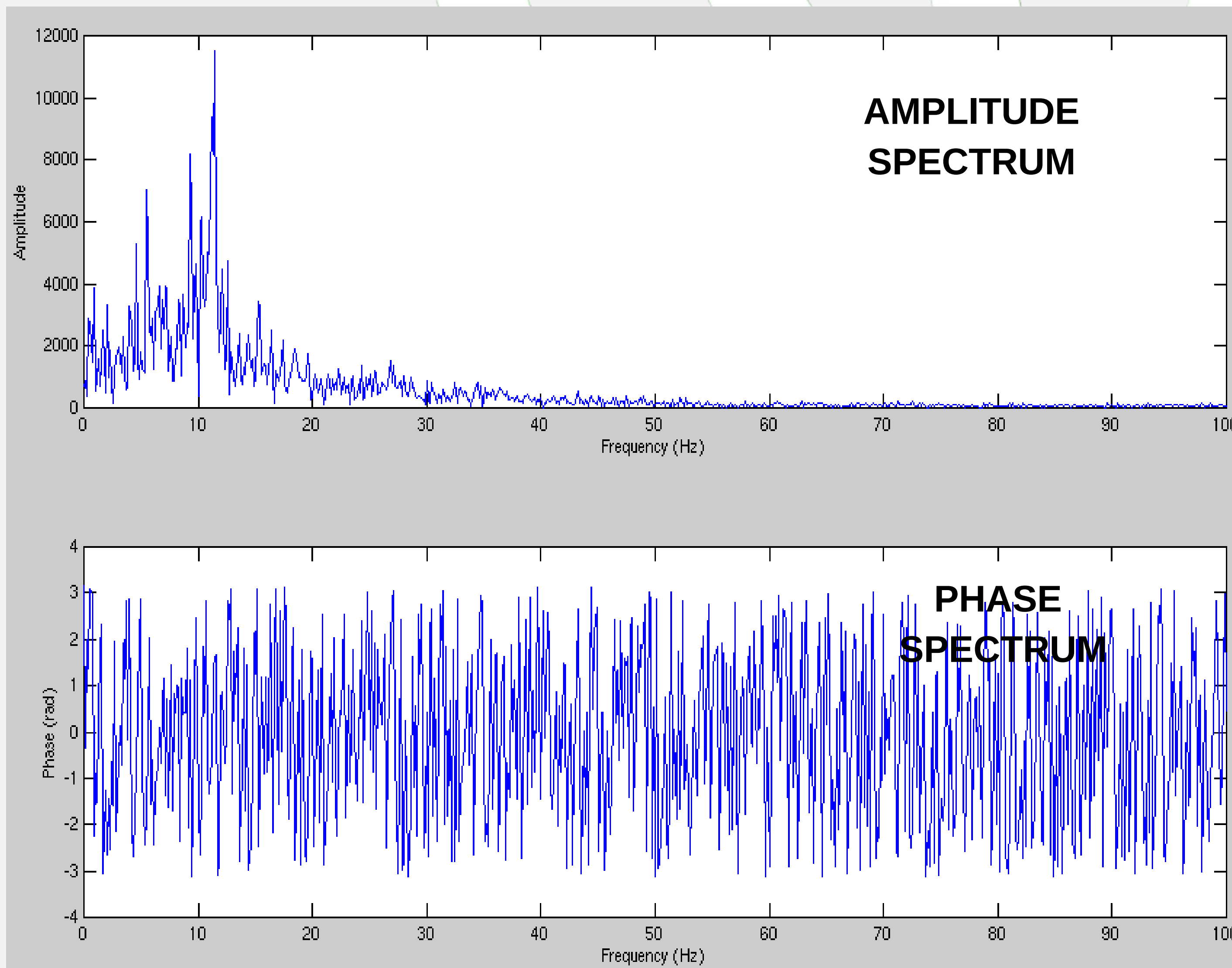


From: Yilmaz (2001)

Example: Seismic trace

seismic trace in time | **and its frequency spectrum**





Fourier Transform Properties (1)

$$S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi f t} dt$$

Direct Fourier Transform

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi f t} df$$

Inverse Fourier Transform

S(f) and s(t) Fourier Transform pairs if

$$s(t) \leftrightarrow S(f)$$

Fourier Transform Properties (2)

1) Linearity: $$a_1 s_1(t) + a_2 s_2(t) \leftrightarrow a_1 S_1(f) + a_2 S_2(f)$$

2) Symmetry: $$s(-t) \leftrightarrow S(-f)$$

$$s^*(t) \leftrightarrow S^*(-f)$$

If s(t) is real: $$S(f) = S^*(-f)$$

i.e., real part and modulus are even functions, imaginary part and phase are odd functions (complex conjugate symmetry)

If s(t) is real and even $\Rightarrow$ S(f) is real and even

If s(t) is real and odd $\Rightarrow$ S(f) is imaginary and odd

Fourier Transform Properties (3)

3) Values at the origin:

$$S(0)=\int_{-\infty}^{\infty} s(t)\,dt$$

$$s(0)=\int_{-\infty}^{\infty} S(f)\,df$$

4) Duality: if $s(t) \leftrightarrow S(f)$

then $S(-t) \leftrightarrow s(f)$

5) Scaling:

$$s(at) \leftrightarrow \frac{1}{|a|} S\left(\frac{f}{a}\right)$$

Particular case: a=-1 $\Rightarrow$ $s(-t) \leftrightarrow S(-f)$

Fourier Transform Properties (4)

6) Time shift:

$$s(t-t_0) \leftrightarrow S(f)\,e^{-j2\pi f t_0}$$

7) Frequency shift:

$$s(t)\,e^{j2\pi f_0 t} \leftrightarrow S(f-f_0)$$

8) Time derivative:

$$\frac{ds(t)}{dt} \leftrightarrow j2\pi f\,S(f)$$

Frequency derivative:

$$t\,s(t) \leftrightarrow \frac{j}{2\pi}\frac{dS(f)}{df}$$

Derivative of order n:

$$\frac{d^n s(t)}{dt^n} \leftrightarrow (j2\pi f)^n S(f)$$

$$t^n s(t) \leftrightarrow \left(\frac{j}{2\pi}\right)^n \frac{d^n S(f)}{df^n}$$

Fourier Transform Properties

9) Frequency multiplication $\Rightarrow$ Time convolution

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \leftrightarrow X(f)H(f)$$

9) is a very useful properties to evaluate the response of a LTI system

10) Time multiplication $\Rightarrow$ Frequency convolution

$$x(t)h(t) \leftrightarrow \int_{-\infty}^{\infty} X(\xi)H(f-\xi)\,d\xi$$

Fourier Transform Properties

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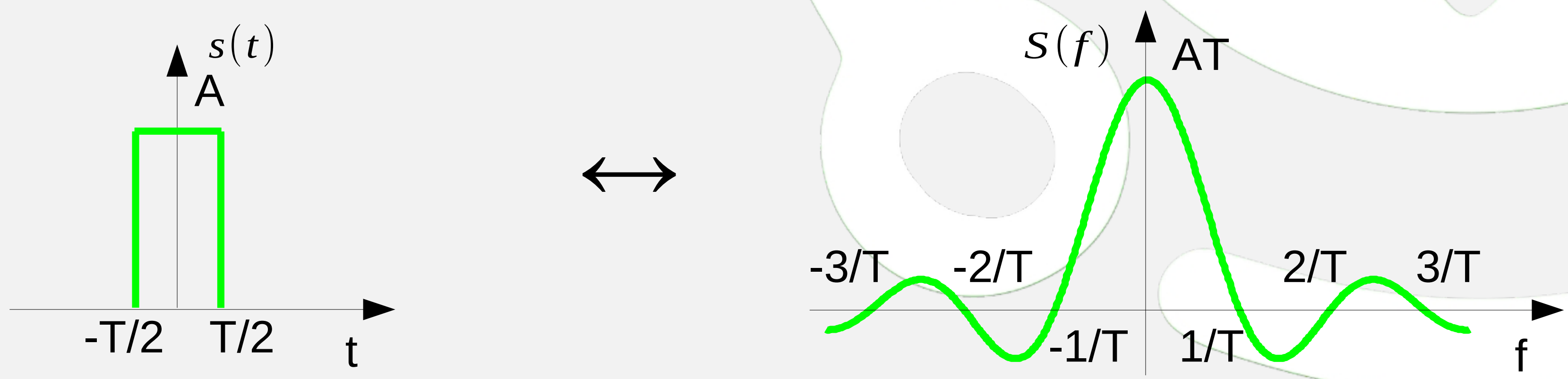
12) Energy spectral density. From Parseval theorem for x(t)=h(t):

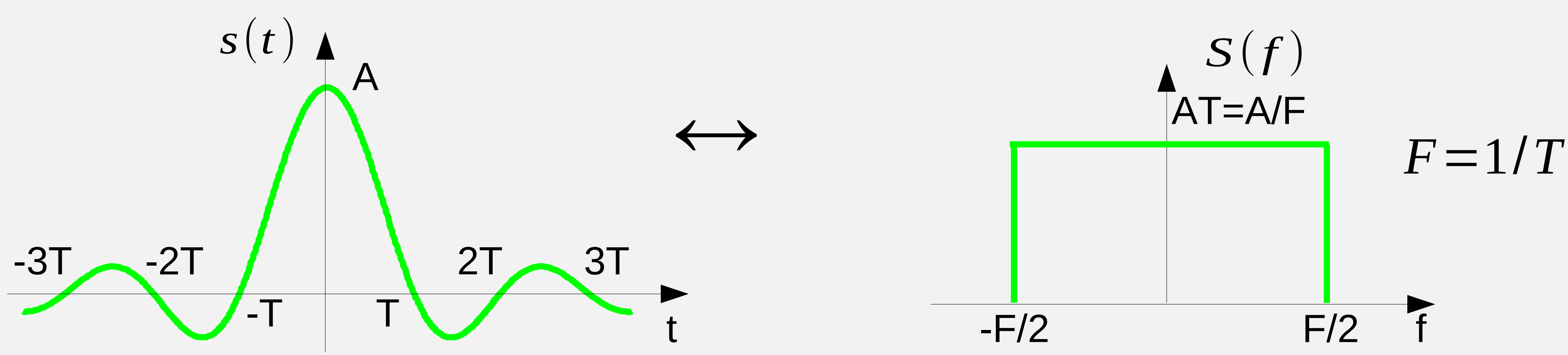
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

E	Signal energy
$\lvert X(f)\rvert^2 df$	Represents the signal energy in an infinitesimal interval *df*
$\lvert X(f)\rvert^2$	is called: energy spectral density

Duality Property

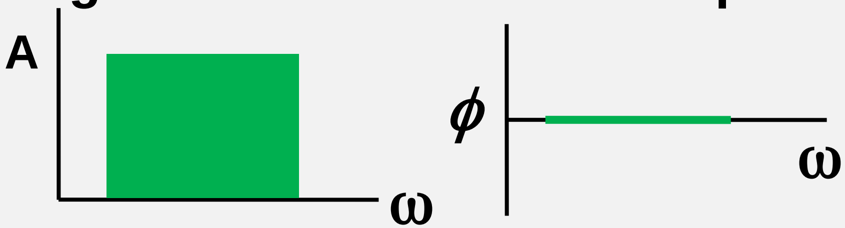
Note: due to the ***Duality Property*** of the Fourier Transform, which relates to the fact that the direct and inverse equations look almost identical except for a factor of 1/T and for a minus sign in the exponential in the integral, the Fourier transform of the Fourier transform is proportional to the original signal reversed in time. That is whenever we have a transform pair, there is a dual pair with the time and frequency variables interchanged.
E.g. a boxcar function in time yields a sinc in frequency and vice-versa

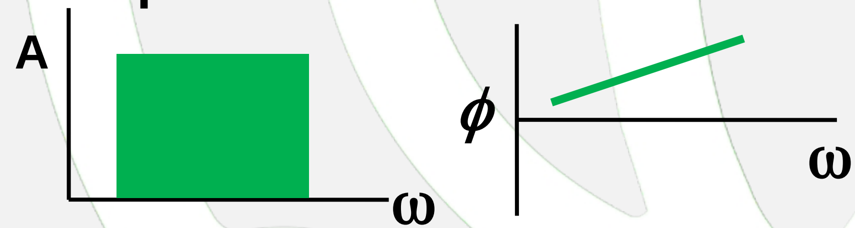


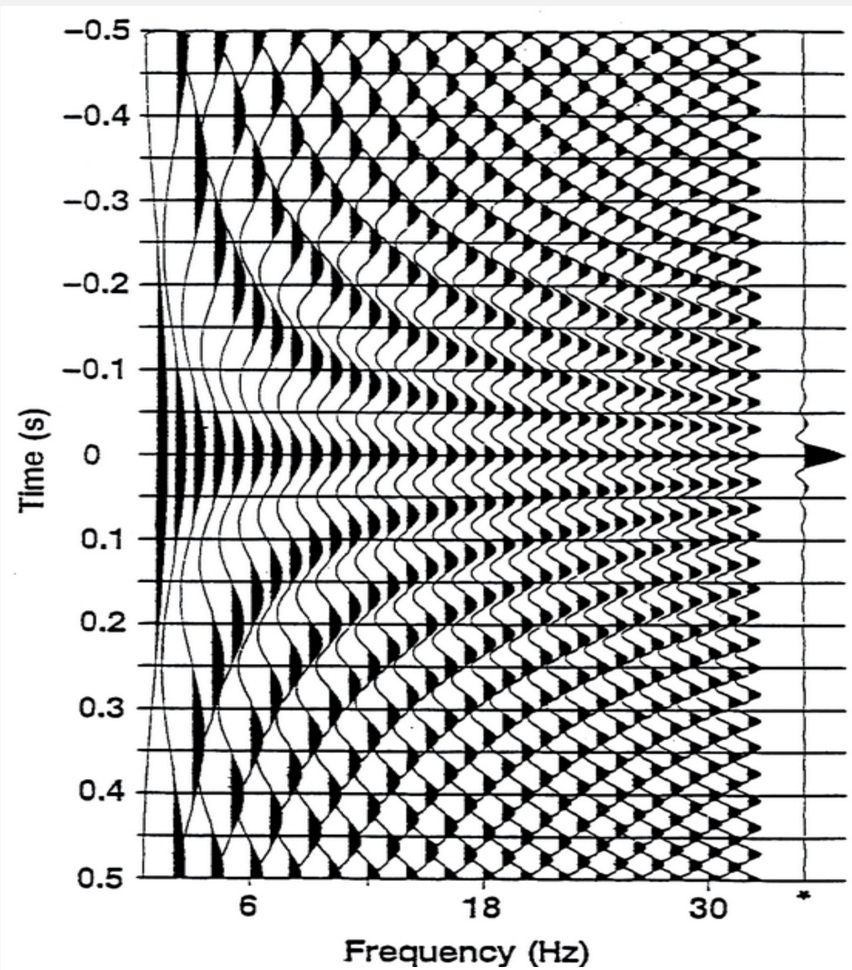


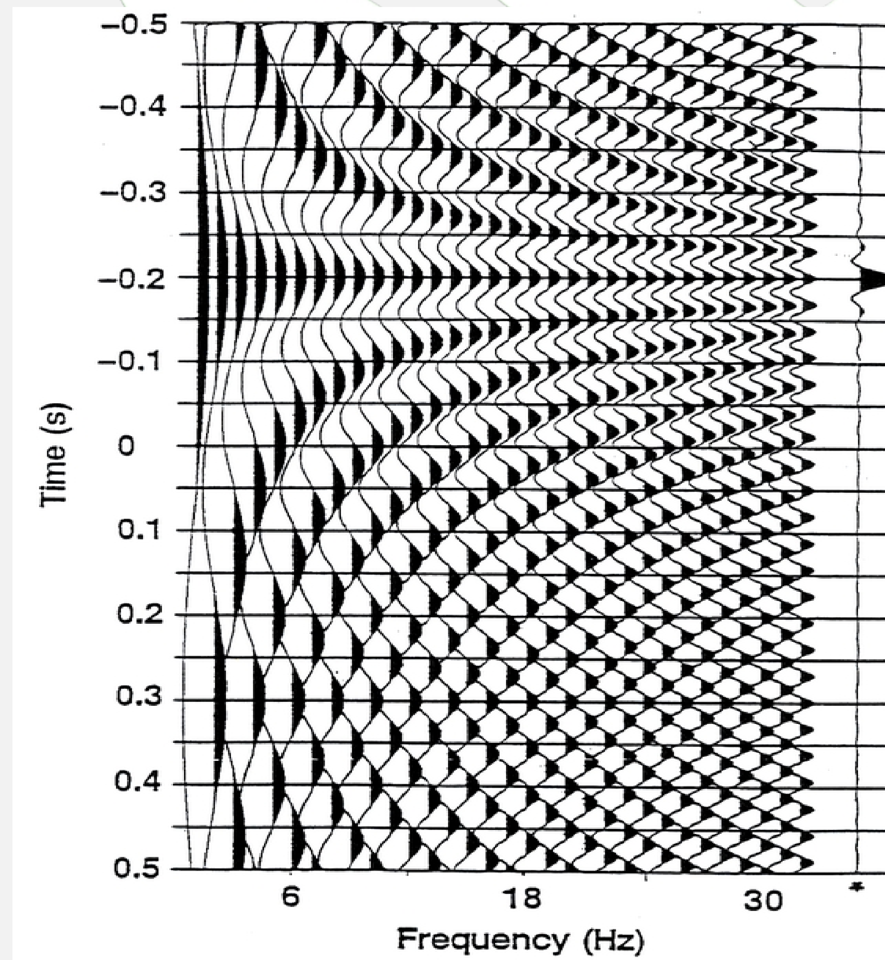
Phase Spectrum

The significance of the Phase Spectrum: Temporal shift ⟷ Phase rotation



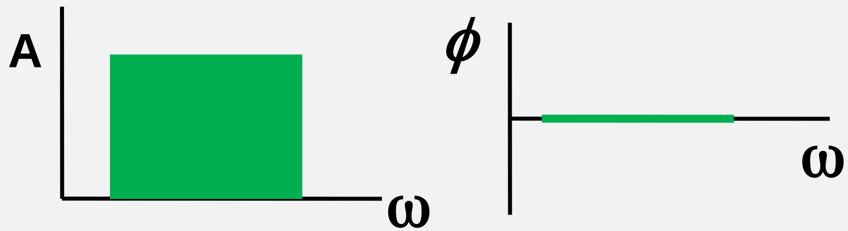


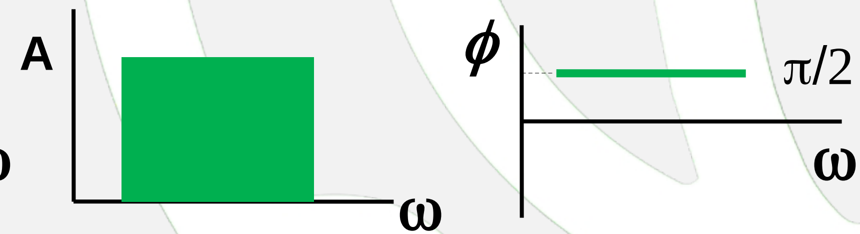


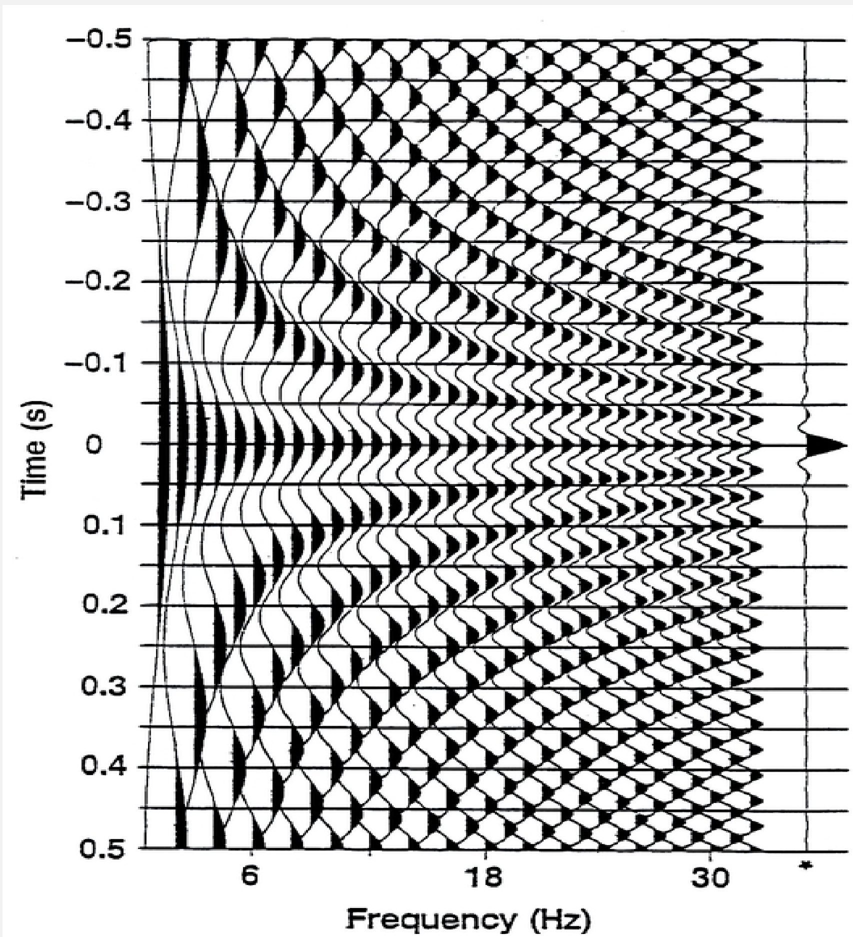


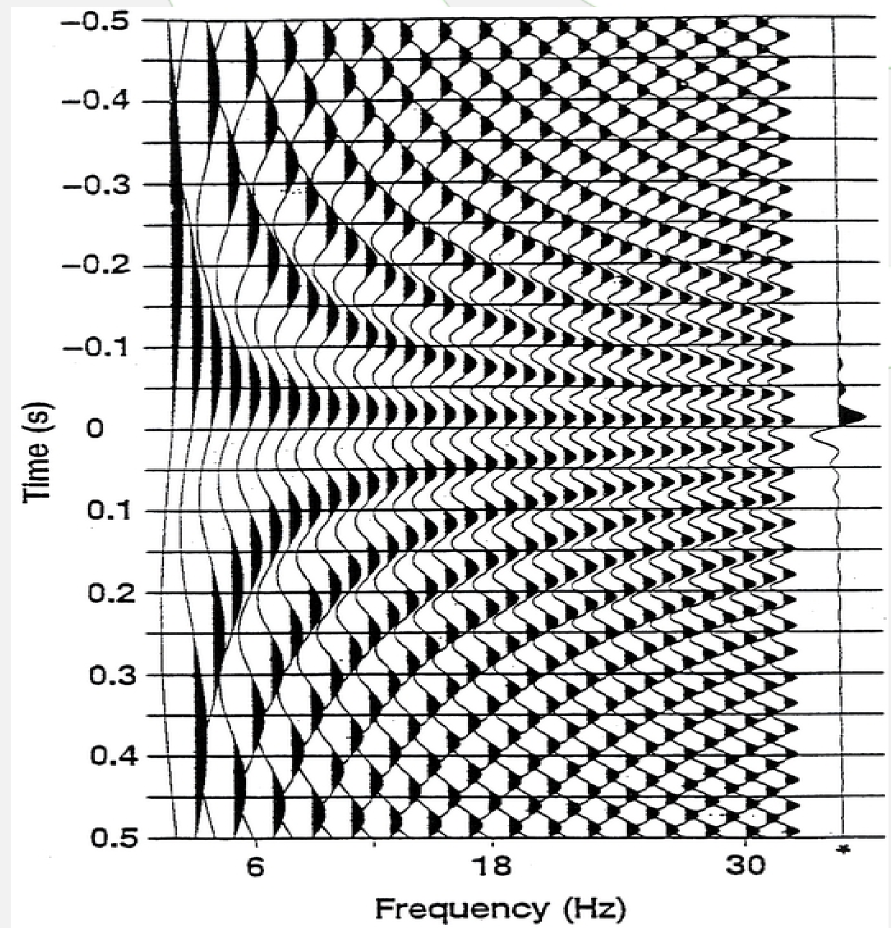
Phase Spectrum

The significance of the Phase Spectrum: Phase shift ⟷ Shape change



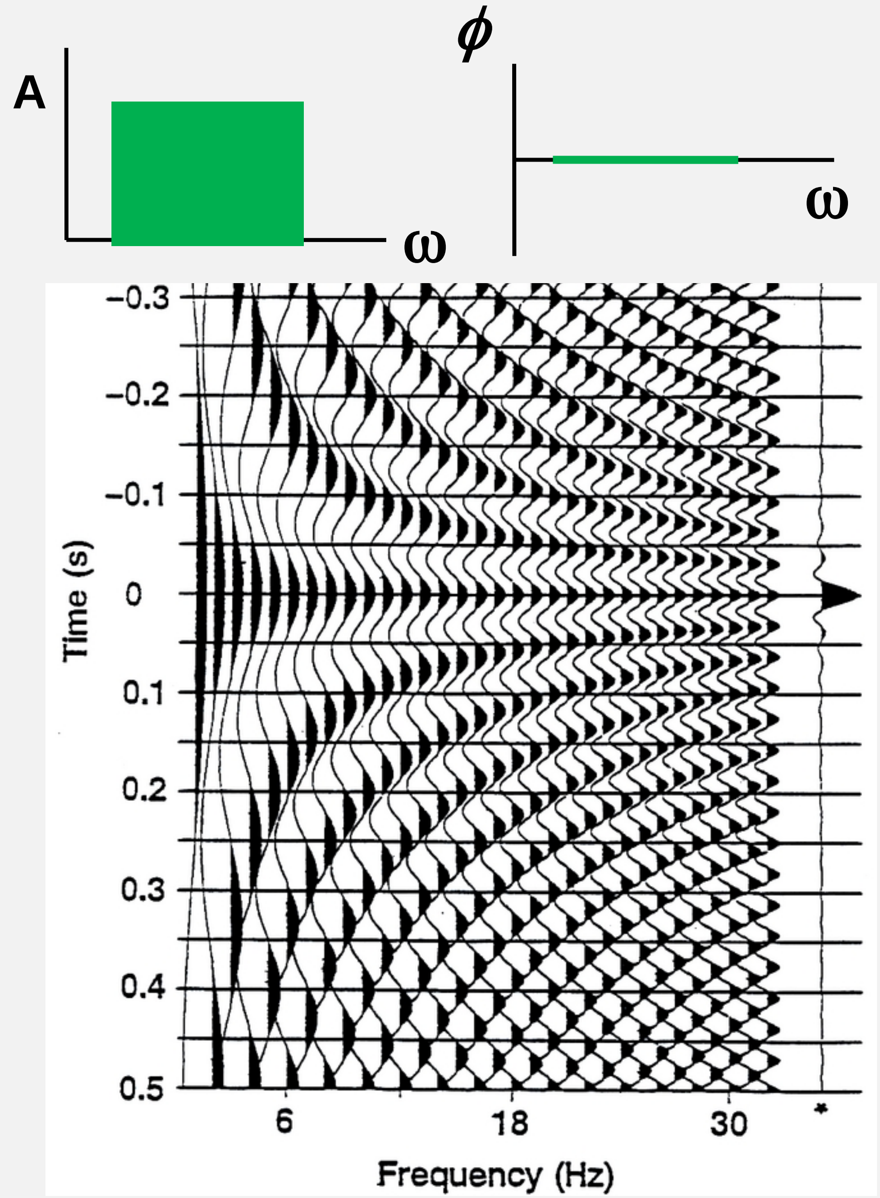


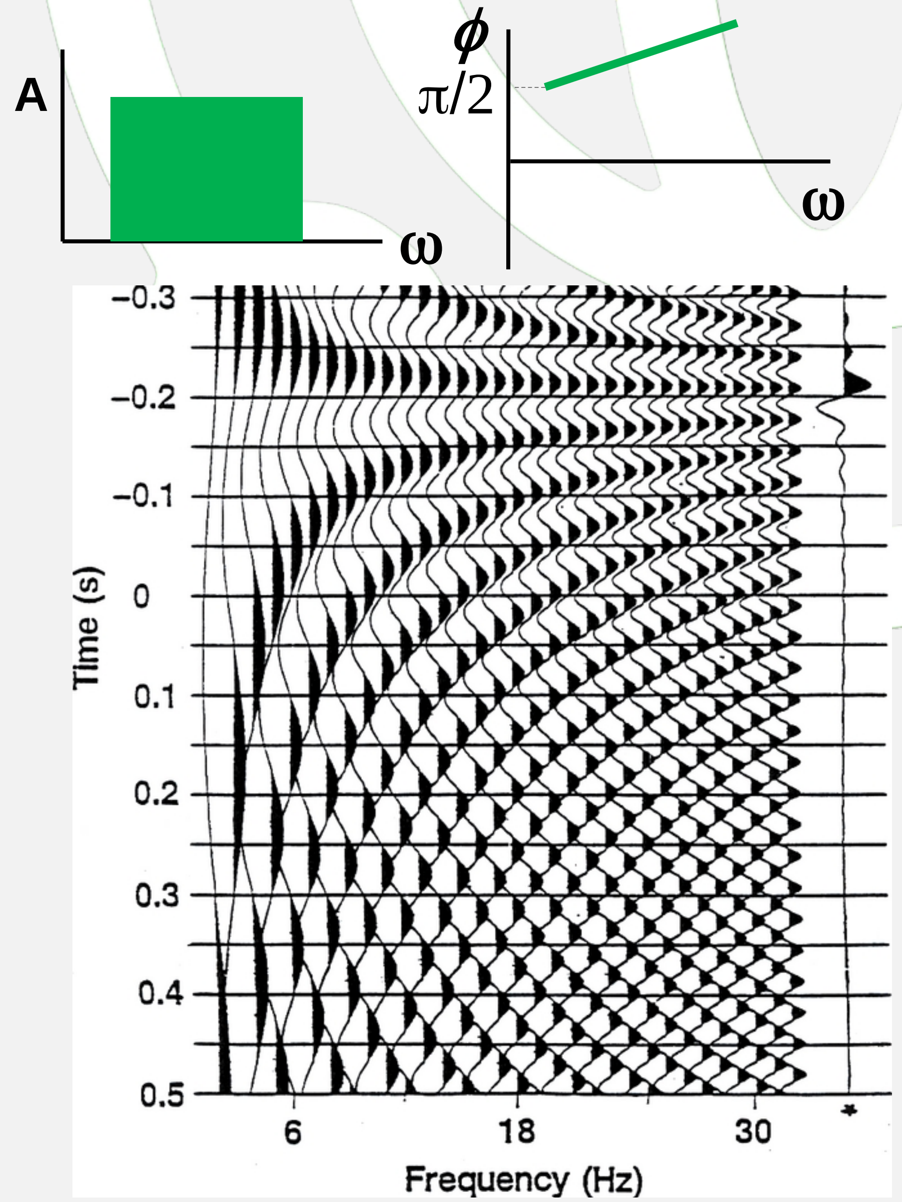




Phase Spectrum

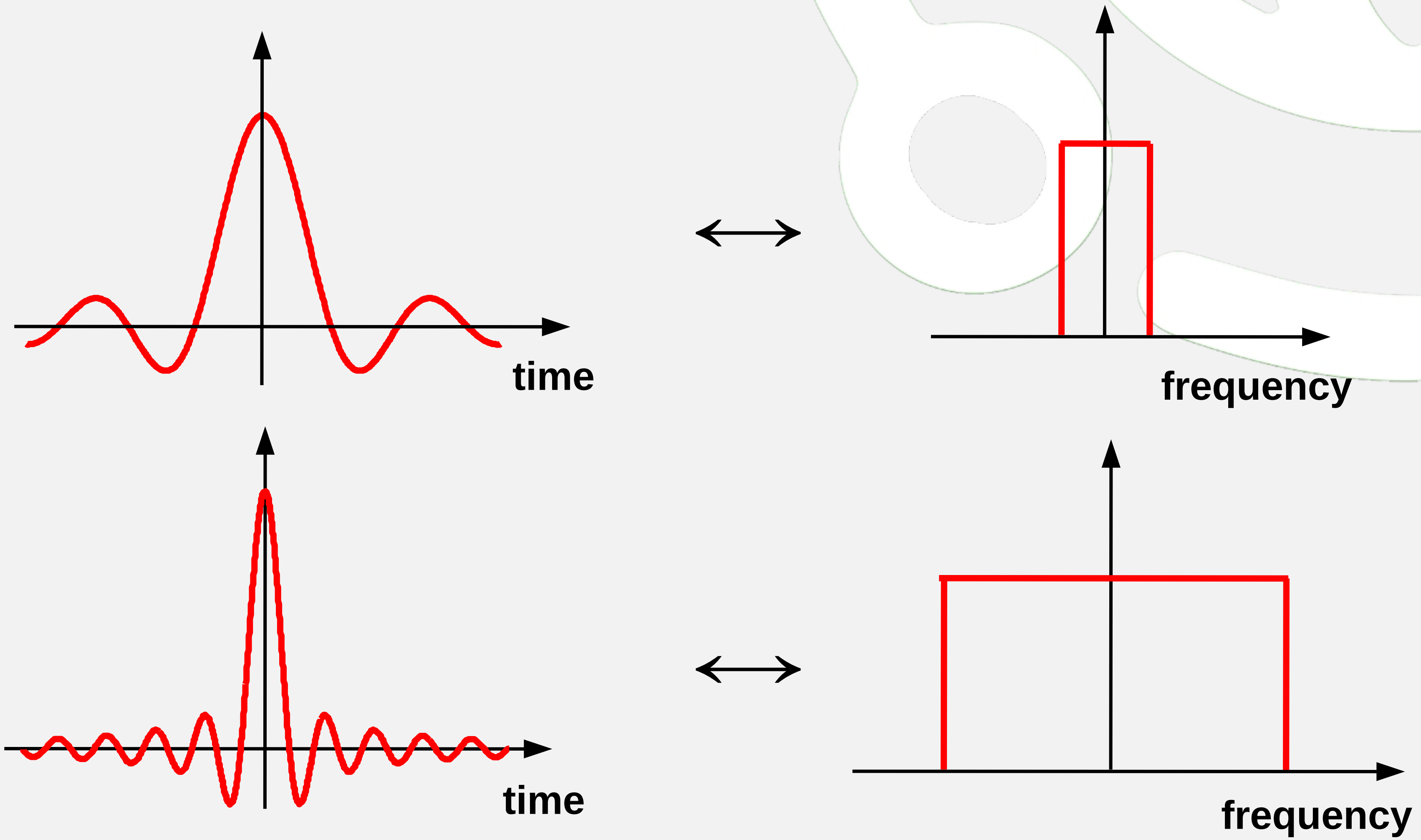
Rotation and Shift of the Phase Spectrum





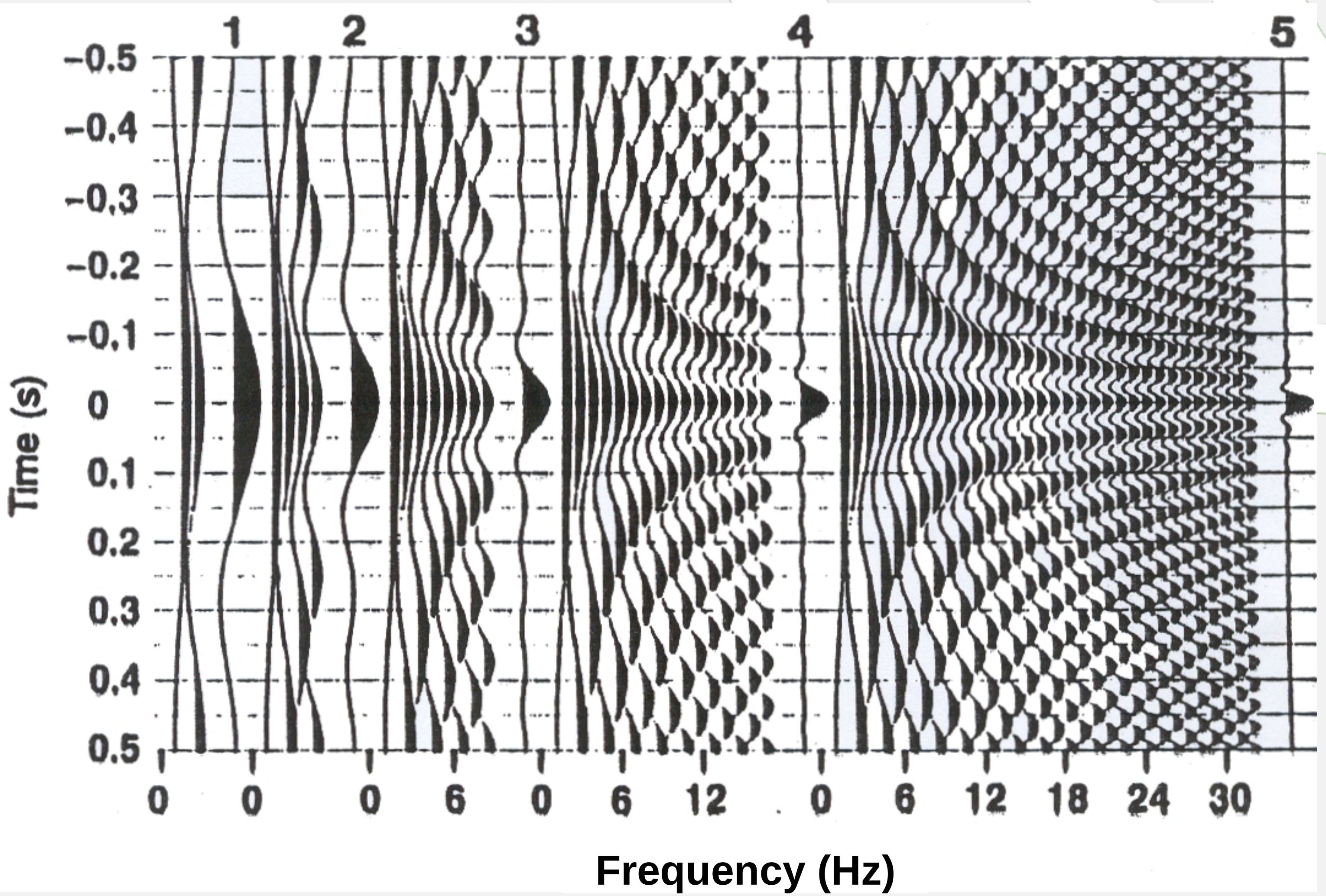
Qualitative relations between time signals and their frequency spectra

Short time signals have a wider frequency band than long wavelets



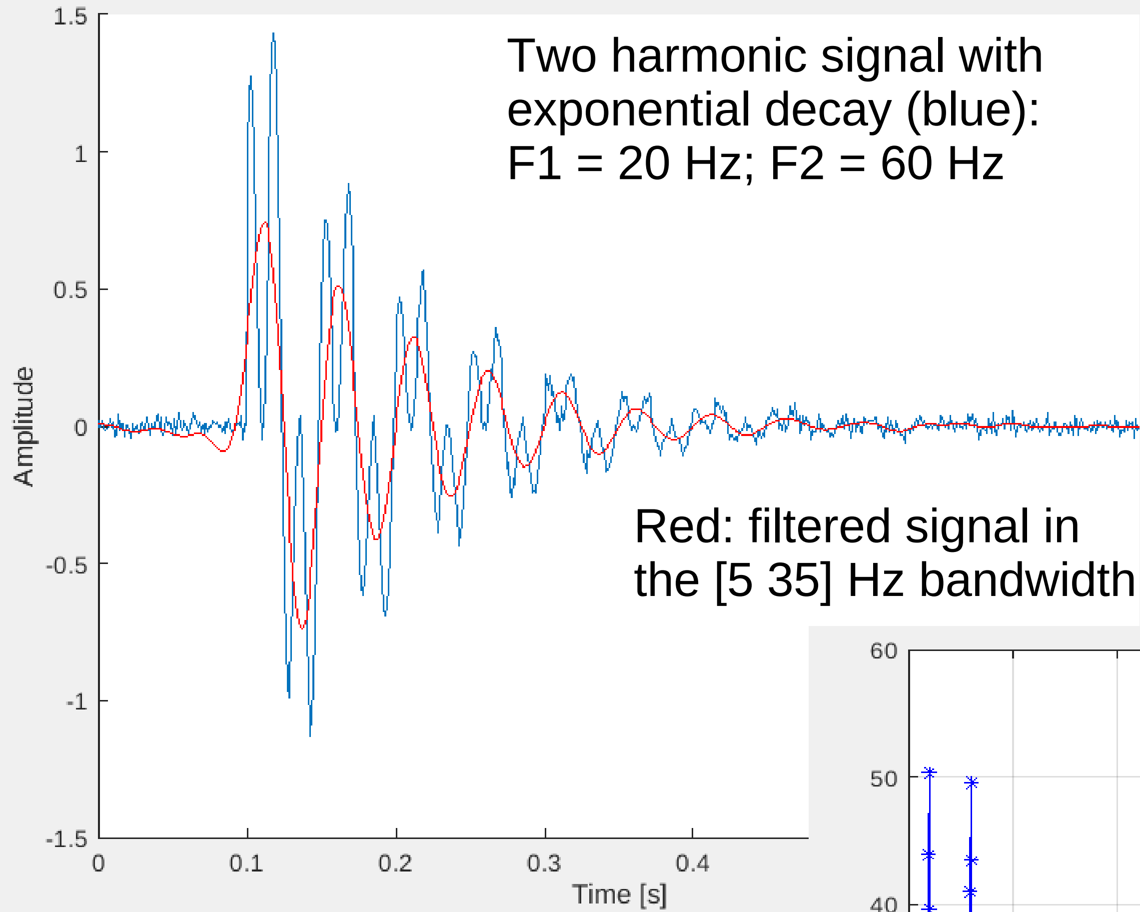
Qualitative relations between time signals and their frequency spectra

Shorter wavelets are composed by more harmonics (here all the harmonics are cosines, that is zero-phase harmonics)

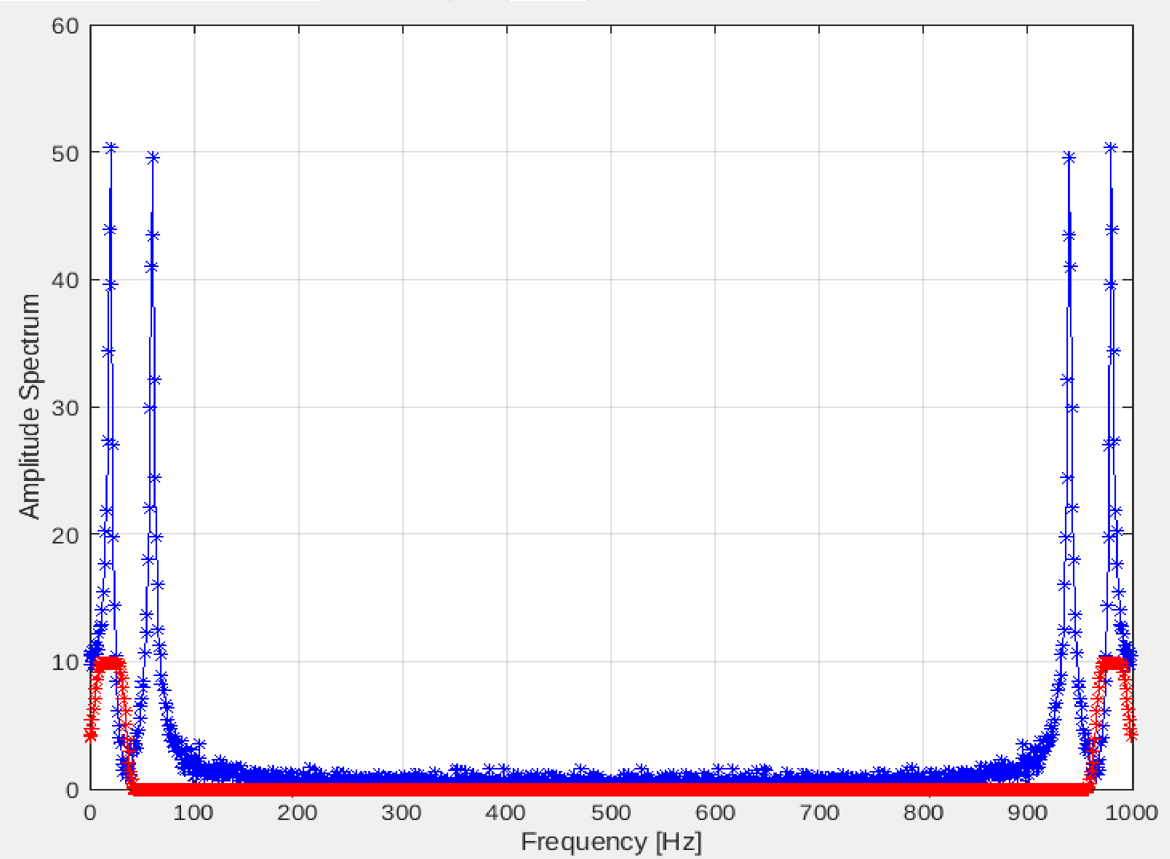


From: Yilmaz (2001)

Example of application of a band-pass filter



Amplitude spectra:
Blue: original spectrum
Red: filter frequency response (multiplied by 10)



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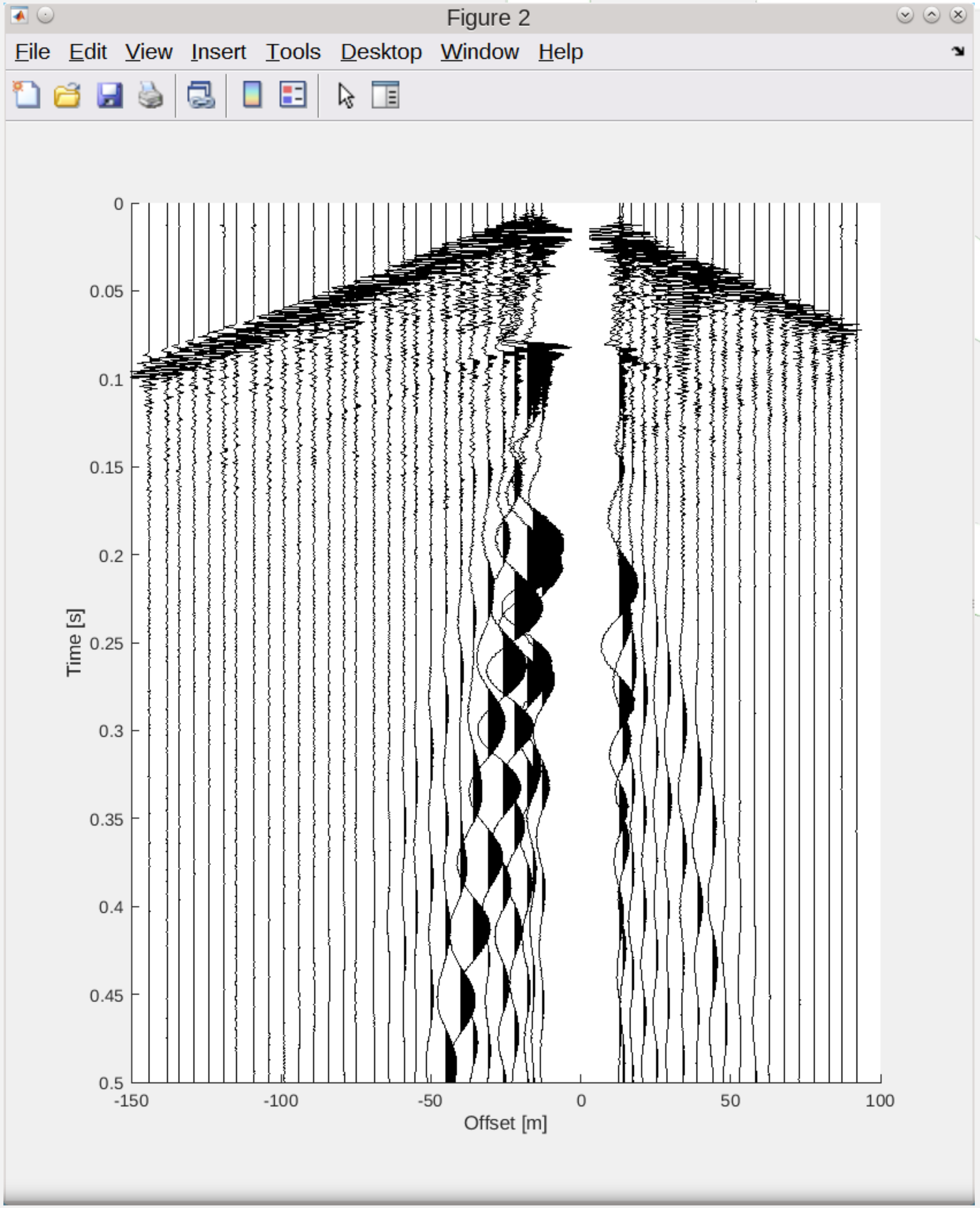
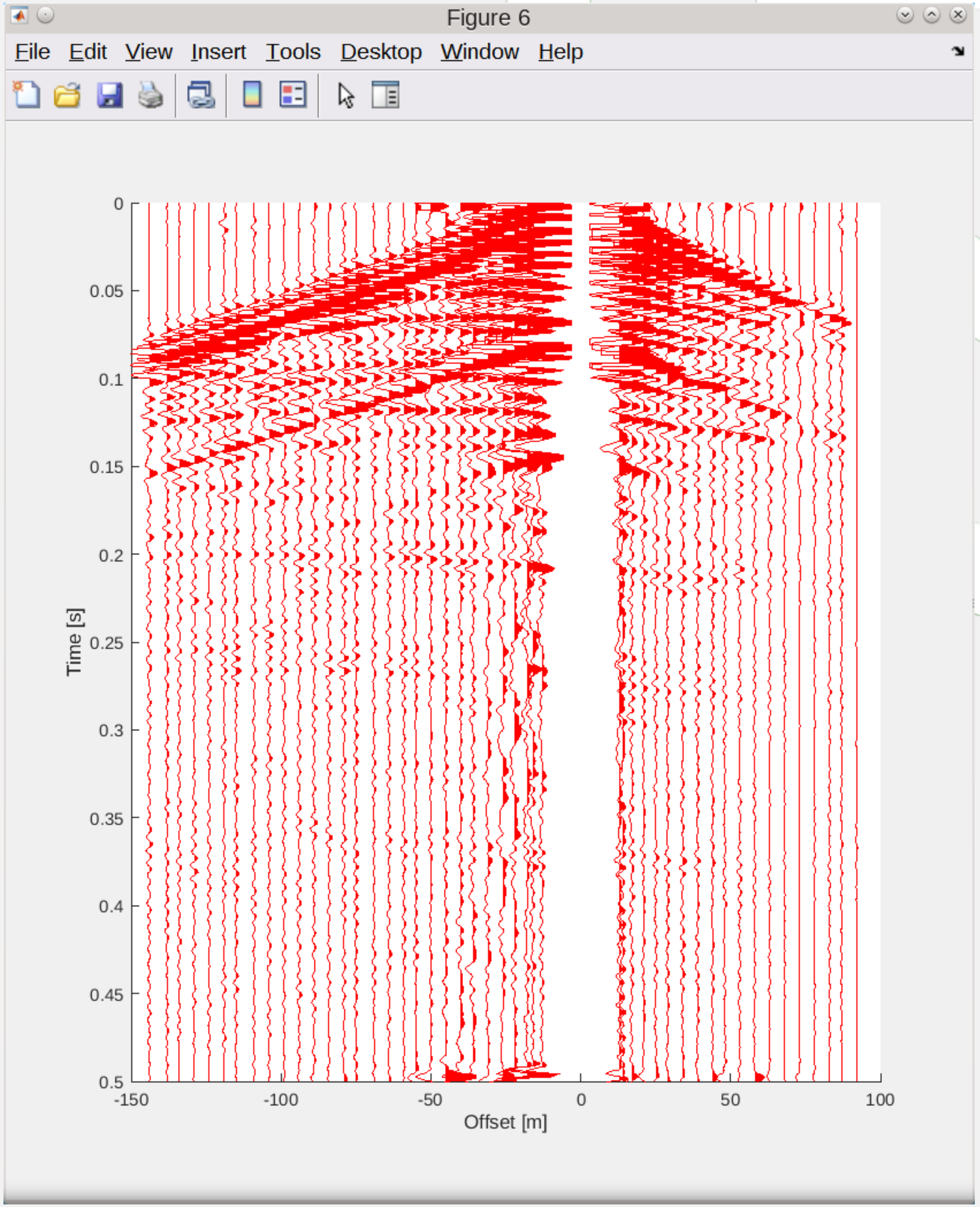


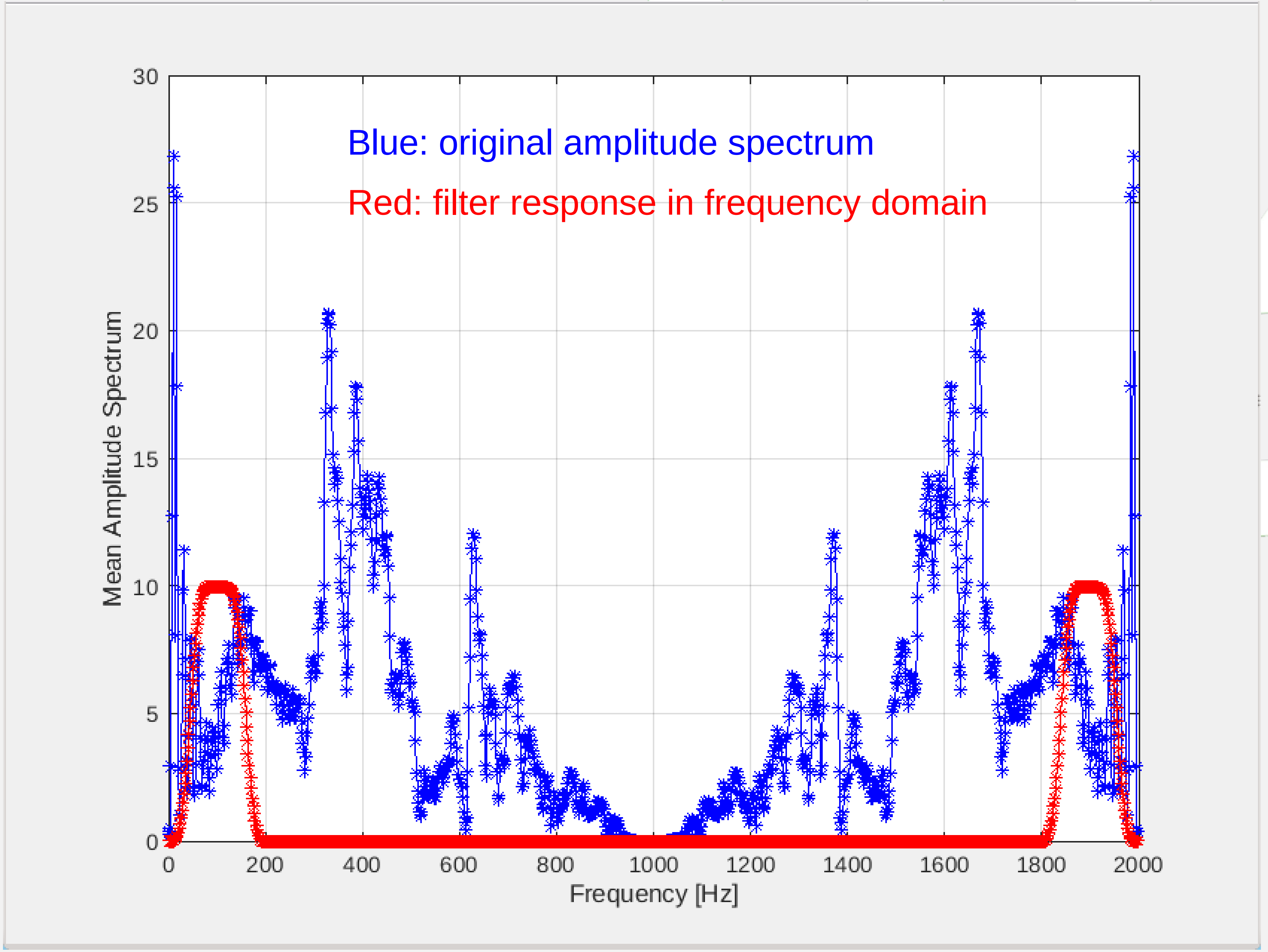
Before filtering

Example of application of a band-pass filter



After filtering

Filter Bandwidth Selection



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